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MATH 5763 — Fall 2015

HW 9

Due: 11/24/2015

Problem 1

Ans:

(a) if $\{B(t) > m\}$, then $B(s) = m, s \in [0,t]$, according to the definition of T_m, which means that $T_m \in [0,t]$. So if event $\{B(t) > m\}$ occures, then $\{T_m \le t\}$ must occures. That also the resaon why $\mathbb{P}(T_m \le t, B(t) > m) = \mathbb{P}(B(t) > m)$.

(b)

$$\begin{aligned}
\mathbb{P}(T_m \le t) &= \mathbb{P}(|B(t)| > m) \\
&= 2\frac{1}{\sqrt{2\pi t}}\int_m^\infty e^{-\frac{x^2}{2t}}dx \\
&= \sqrt{\frac{2}{\pi t}}\int_m^\infty e^{-\frac{x^2}{2t}}dx \Leftarrow \left(x = m\sqrt{\frac{t}{y}}\right) \\
&= \int_0^t \frac{|m|}{\sqrt{2\pi y^3}}e^{-m^2/(2y)}dy
\end{aligned}$$

(c) From above equations we known the c.d.f. of T_m. so the p.d.f. of T_m is $\frac{|m|}{\sqrt{2\pi y^3}}e^{-m^2/(2y)}$

Problem 2

Ans:

(a)

$$\begin{aligned}
R_Z(t,s) &= \int_0^s\int_0^t \min\{u,v\}dudv \\
&= \int_0^s \left(\int_0^v udu + \int_v^t vdu\right)dv \\
&= \int_0^s \left(\frac{v^2}{2} + v(t-v)\right)dv \\
&= \left(\frac{v^3}{6} + \frac{v^2 t}{2} - \frac{v^3}{3}\right)\Big|_0^s \\
&= s^2(\frac{t}{2} - \frac{s}{6})
\end{aligned}$$

(b)

$$\begin{aligned}
Cov[Z(t)-Z(s), Z(s)] &= \mathbb{E}[Z(t)-Z(s), Z(s)] - \mathbb{E}[Z(t)-Z(s)]\,\mathbb{E}[Z(s)] \\
&= \mathbb{E}[Z(t)Z(s)] - \mathbb{E}[Z^2(s)] - 0 \Leftarrow \mathbb{E}[Z(s)] = 0 \\
&= s^2(\frac{t}{2} - \frac{s}{6}) - \frac{s^3}{3} \\
&= \frac{s^2 t}{2} - \frac{s^3}{2}
\end{aligned}$$

(c) These two intervals are not disjointed, so they are not independent.

HW 9

Problem 3

Ans: For $W(t)$, Let $\{F(t); t \geq 0\}$ be a natural filtration and $s < t$.

$$\begin{aligned}
\mathbb{E}[W(t+s)|F(s)] &= \mathbb{E}[W(t+s)|W(s)] \\
&= \mathbb{E}[W(t+s) - W(s) + W(s)|W(s)] \\
&= \mathbb{E}[W(t+s) - W(s)|W(s)] + W(s) \\
&\Rightarrow \text{According to Gaussian increasement property } \mathbb{E}[W(t+s) - W(s)|W(s)] = 0 \\
&= W(s)
\end{aligned}$$

Hence, $W(t)$ is a martingales.

For $W(t)^2 - t$, Let $\{F(t); t \geq 0\}$ be a natural filtration and $s < t$.

$$\begin{aligned}
\mathbb{E}[W(t)^2 - t|F(s)] &= \mathbb{E}[(W(t) - W(s) + W(s))^2 - t|F(s)] \\
&= \mathbb{E}[(W(t) - W(s))^2] + W(s)^2 + 2W(S)\,\mathbb{E}[W(t) - W(s)] - t \\
&\Rightarrow \text{We know that}\,\mathbb{E}[W(t) - W(s)] = 0, \mathbb{E}[(W(t) - W(s))^2] = t - s \\
&= t - s + W(s)^2 + 2W(s) * 0 - t \\
&= W(s)^2 - s
\end{aligned}$$

Hence, $W(t)^2 - t$ is a martingales.