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MATH 5763 — Fall 2015 **HW 8** Due: 11/12/2015

Problem 1

Ans:

(a) if $t_1 > a$, then he can corss the street without waiting for the first car, So the total time he cost is $T = a$. However if $t_1 < a$, then he must wait until the first car passed, then depends on the situation, the total time he cost will be $\mathbb{E}[T] = t_1 + \mathbb{E}[T]$.

(b)

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[\mathbb{E}[T|T_1]] \\
&= \int_0^a \mathbb{E}[T|T_1]\,\mathbb{P}(T|T_1 = t_1)dt_1 + \int_a^\infty \mathbb{E}[T|T_1]\,\mathbb{P}(T|T_1 = t_1) \\
&= \int_0^a (t_1 + \mathbb{E}[T])\lambda e^{-\lambda t_1}dt_1 + \int_a^\infty a\lambda e^{-\lambda t_1}dt_1 \\
&= \int_0^a (t_1 + \mathbb{E}[T])\lambda e^{-\lambda t_1}dt_1 + (-ae^{-\lambda t_1})\Big|_a^\infty \\
&= \int_0^a (t_1 + \mathbb{E}[T])\lambda e^{-\lambda t_1}dt_1 + ae^{-\lambda a}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{E}[T] &= \int_0^a (t_1 + \mathbb{E}[T])\lambda e^{-\lambda t_1}dt_1 + ae^{-\lambda a} \\
&= \int_0^a t_1\lambda e^{-\lambda t_1}dt_1 + \int_0^a \mathbb{E}[T]\lambda e^{-\lambda t_1}dt_1 + ae^{-\lambda a} \\
&= \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda a} - ae^{-\lambda a} + \mathbb{E}[T](1 - e^{-\lambda a}) + ae^{-\lambda a}
\end{aligned}$$

By solving the above equation, we got that $\mathbb{E}[T] = \frac{1}{\lambda e^{-a\lambda}} - \frac{1}{\lambda} = \frac{1}{\lambda}(\frac{1}{e^{-a\lambda}} - 1)$

(d)

$$\lim_{\lambda\to 0} \mathbb{E}[T] = a$$
$$\lim_{\lambda\to \infty} \mathbb{E}[T] = \infty$$

It is resonable that $\mathbb{E}[T] = a$ when $\lambda \to 0$, it means that no car passed the street; otherwise, it $\lambda \to \infty$, then he will never get a chance to corss the street, so the $\mathbb{E}[T] = \infty$.

(e) For case(i):

$$\mathbb{E}[T] = \frac{1}{\lambda}(\frac{1}{e^{-a\lambda}} - 1) + \frac{1}{\mu}(\frac{1}{e^{-b\mu}} - 1)$$

For case(ii):

$$\mathbb{E}[T] = \frac{1}{(\lambda + \mu)}\left(\frac{1}{e^{(-a-b)(\lambda+\mu)}} - 1\right)$$

(f) Case (ii) will cost more time then Case(i). In case(i), you need to consider the more short time cost, in Case(ii), a longer time you need to be considered, it seems to more hard to meet the conditions.

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Problem 2

Ans:

(a)

$$\begin{aligned}\sum_{i=0}^{\infty}\delta_i &= \mathbb{E}[e^{-\lambda S}] + \mathbb{E}[(\lambda S)e^{-\lambda S}] + \mathbb{E}[\frac{(\lambda S)^2}{2!}e^{-\lambda S}] + \cdots \\ &= \mathbb{E}[e^{-\lambda S} + \lambda S e^{-\lambda S} + \frac{(\lambda S)^2}{2!}e^{-\lambda S} + \cdots] \\ &= \mathbb{E}[e^{-\lambda S}(1 + \frac{\lambda S}{1!} + \frac{(\lambda S)^2}{2!} + \cdots)] \Leftarrow \text{Taylor series} \\ &= \mathbb{E}[e^{-\lambda S} * e^{\lambda S}] \\ &= 1\end{aligned}$$

So matirx **P** is stochastic matrix.

(b)

$$\begin{aligned}\delta_j &= \int_0^{\infty}\frac{(\lambda S)^j}{j!}e^{-\lambda S}\mu e^{-\mu S}dS \\ &= \frac{\mu}{j!}\int_0^{\infty}(\lambda S)^j e^{-(\lambda+\mu)S}dS \Leftarrow \text{change of variable } x = (\lambda+\mu)S \\ &= \frac{\mu}{j!}\frac{1}{\lambda+\mu}\int_0^{\infty}(\lambda S)^j e^{-x}dx \Leftarrow dS = \frac{1}{\lambda+\mu}dx \\ &= \frac{\mu}{j!}\frac{1}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^j\int_0^{\infty}x^j e^{-x}dx \Leftarrow \int_0^{\infty}x^j e^{-x}dx = j! \\ &= \left(\frac{\lambda}{\lambda+\mu}\right)^j\frac{\mu}{\lambda+\mu}\end{aligned}$$

(c)

$$\begin{aligned}\delta_j &= \frac{(\lambda\mu^{-1})^j}{j!}e^{-\lambda\mu^{-1}} \\ &= \frac{(\frac{\lambda}{\mu})^j}{j!}e^{-\frac{\lambda}{\mu}}\end{aligned}$$

(d)

$$\begin{aligned}\delta_j &= \int_0^{\frac{2}{\mu}}\frac{(\lambda S)^j}{j!}e^{-\lambda S}\frac{\mu}{2}dS \\ &= \frac{\mu}{2j!}\int_0^{\frac{2}{\mu}}(\lambda S)^j e^{-\lambda S}dS \\ &= \frac{\mu}{2j!} * \gamma(j+1, \frac{2}{\mu}) \Leftarrow \text{Incomplete Gamma Function}\end{aligned}$$

Problem 3

Ans:

(a)

$$\begin{aligned} p_j(t) &= \mathbb{P}(X_t = j) = \sum_{i=0}^{1} \mathbb{P}(X_t = j | X_0 = i)\, \mathbb{P}(X_0 = i) \\ &= \sum_{i=0}^{1} p_i(0) p_{ij}(t) \\ p_0(t) &= \frac{1}{2} * \frac{1}{2}(1 + e^{-2\lambda t}) + \frac{1}{2}\frac{1}{2}(1 - e^{-2\lambda t}) = \frac{1}{2} \\ p_1(t) &= \frac{1}{2} * \frac{1}{2}(1 - e^{-2\lambda t}) + \frac{1}{2}\frac{1}{2}(1 + e^{-2\lambda t}) = \frac{1}{2} \end{aligned}$$

Because $\mathbf{p}(0)$ is the stationary distribution of the Chain X, which means that $p_0 P_t = p_0$. So it's obvious that $\mathbf{p}(0) = \mathbf{p}(t)$.

(b)

$$\begin{aligned} \mathbb{E}[X_t] &= \sum_{i=0}^{1} i * p_i(t) = 1 * \frac{1}{2} = \frac{1}{2} \\ \mathbb{E}[X_s X_{s+t}] &= \mathbb{E}[\mathbb{E}[X_s X_{s+t} | X_s]] \\ &= \sum_{k=0}^{k=1} \mathbb{E}[X_s X_{s+t} | X_s = k]\, \mathbb{P}(X_s = k) \\ &= \mathbb{P}(X_s = 1)\, \mathbb{E}[X_s X_{s+t} | X_s = 1] \\ &= \frac{1}{2} * 1 * p_{11}(t) = \frac{1}{4}(1 + e^{-2\lambda t}) \end{aligned}$$

(c) Yes. first $\mathbb{E}[X_t] = \frac{1}{2}$, second $Var(X) < \infty$. and $\mathbb{E}[X_s, X_{s+t}]$ only depend on $(s+t) - s$.

(d)

$$\begin{aligned} C_X(s, s+t) &= R_X(s, s+t) - m_X(s) m_X(s+t) \\ &= \frac{1}{4}(1 + e^{-2\lambda t}) - \mathbb{E}[X_t]\, \mathbb{E}[X_{t+s}] \Leftarrow \mathbb{E}[X_{t+s}] = \mathbb{E}[X_t] = \frac{1}{2} \\ &= e^{-2\lambda t} \end{aligned}$$

(e)

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[\int_0^t X_u d_u\right] = \int_0^t \mathbb{E}[X_u]du \\
&= \frac{u}{2}\bigg|_0^t = \frac{t}{2} \\
\mathbb{E}[Y_t^2] &= 2\int_0^t \left(\int_0^v \mathbb{E}[X_u X_v]du\right)dv \\
&= \frac{1}{2}\int_0^t \int_0^v (1 + e^{-2\lambda(v-u)})dudv \\
&= \frac{1}{2}\int_0^t \left[u + \frac{e^{-2\lambda(v-u)}}{2\lambda}\bigg|_0^v\right]dv \\
&= \frac{1}{2}\int_0^t \left(v + \frac{1}{2\lambda} - \frac{e^{-2\lambda v}}{2\lambda}\right)dv \\
&= \frac{1}{2} * \frac{1}{2}v^2 + \frac{v}{2\lambda} + \frac{e^{-2\lambda v}}{4\lambda^2}\bigg|_0^t \\
&= \frac{t^2}{4} + \frac{t}{4\lambda} + \frac{e^{-2\lambda t}}{8\lambda^2} - \frac{1}{8\lambda^2}
\end{aligned}
$$

(f)

$$
\begin{aligned}
Var(Y_t) &= \mathbb{E}[Y_t^2] - (\mathbb{E}[Y_t])^2 \\
&= \frac{t}{4\lambda} + \frac{e^{-2\lambda t}}{8\lambda^2} - \frac{1}{8\lambda^2} \\
\lim_{t \to 0^+} Var(Y_t) &= 0
\end{aligned}
$$