Name:Chao Jiang
MATH 5763 — Fall 2015 **HW 7** Due: 10/30/2015

Problem 1

Let $\{N_t : t \geq 0\}$ be a Poisson process with constant rate λ. Define the continuous time processes $\{K_t : t \geq 0\}$, $\{L_t : t \geq 0\}$, $\{M_t : t \geq 0\}$ by

$$K_t = N_{t+2} - N_2, \quad L_t = N_{2t}, \quad M_t = N_{\sqrt{t}}$$

Which of these processes is/are also a constant-rate Poisson process? What are their rates? Justify your answers briefly.

Hint: It is easy to find the distributions of the random variables K_t, L_t, and M_t, since you know that N_t has Poisson distribution with parameter λ_t.

Ans: We know that Poisson Process has independnt increments $N(\tau + t) - N(\tau) \sim Poi(\lambda t)$. So,

$$\begin{aligned} K_t &= N_{t+2} - N_2 = N_{(t+2)-2} - N_{2-2} = N_t - N_0 = N_t \sim Poi(\lambda) \\ L_t &= N_{2t} \sim Poi(2\lambda) \Longleftarrow \text{depends on the length of the increment} \end{aligned}$$

Because the Poisson process has increments that have a distribution that is Poisson and only depends on the length of the increment.

The increments of M_t are not stationary. For instance, $9 - 0 = 25 - 16$ But,

$$\begin{aligned} \mathbb{P}(M_0 - M_0 = 1) &= \mathbb{P}(N_3 - N_0 = 1) = \mathbb{P}(N_3 = 1) = \frac{(\lambda * 3)^1}{1!} e^{-\lambda 3} = 3\lambda e^{-3\lambda} \\ \mathbb{P}(M_{25} - M_{16} = 1) &= \mathbb{P}(N_5 - N_4 = 1) = \mathbb{P}(N_1 = 1) = \frac{(\lambda * 1)^1}{1!} e^{-\lambda 1} = \lambda e^{-\lambda} \end{aligned}$$

So, even though M_t has independent increment, it is not a constant-rate Poisson Process.

Problem 2

A system is made up of two components. We suppose that the lifetime (in years) of each component has an exponential distribution with parameter $\lambda = 2$ year^{-1} , and that the components operate independently. When the system goes down, the two components are then immediately replaced by new ones. Consider the following three cases:

I. the two components are placed in series (so that both components must function for the system to work);

II. the two components are placed in parallel (so that a single operating component is sufficient for the system to function) and the two components operate at the same time);

III. the two components are placed in parallel, but only one component operates at a time, and the other component is in standby (i.e., ready to replace the first component when it fails).

Let $\{N_t : t \geq 0\}$ be the number of system failures in the interval $[0, t]$. Answer the following questions in each of the cases above.

(a) Is $\{N_t : t \geq 0\}$ a Poisson process? If it is, what is its rate? If it is not, justify, and determine the probability distribution of the inter-event times τ_j.

(b) What is the average time elapsed between two consecutive system failures? In two of the above three cases the answer is obvious (but I do want to see your calculations). Please discuss your results in these two cases.

Ans: Let $\tau_1 \sim Exp(\lambda), \tau_2 \sim Exp(\lambda)$ be independent. RVs equal to the lifetime of the two components, with $\lambda = 2yr^{-1}$. Let T be the lifetime of the Whole system.

I. Clearly, in this case, $T = \min(\tau_1, \tau_2)$,Using that

$$F_{\tau_j}(t) = \begin{cases} 0 & t < 0. \\ 1 - e^{-\lambda t} & t > 0. \end{cases}$$

we obtain for $t > 0$

$$\begin{aligned} F_T(t) &= \mathbb{P}(T \leq t) = \mathbb{P}(\min(\tau_1, \tau_2) \leq t) \\ &= 1 - \mathbb{P}(\min(\tau_1, \tau_2) > t) \\ &= 1 - \mathbb{P}(\tau_1 > t, \tau_2 > t) \leftarrow independent \\ &= 1 - \mathbb{P}(\tau_1 > t)\,\mathbb{P}(\tau_2 > t) \\ &= 1 - [1 - F_{\tau_1 > t}(t)][1 - F_{\tau_2 > t}(t)] \\ &= 1 - (e^{-\lambda t})^2 = 1 - e^{-2\lambda t} \end{aligned}$$

So $T \sim Exp(2\lambda)$, Since the inter-event times T of the process N (the moments when the whole system stops functioning) are exponential RVs with parameter 2λ, the process N is Poisson with rate 2λ. The average time between two consecutive system failures is $\mathbb{E}[T] = \frac{1}{2\lambda} = \frac{1}{2*2yr^{-1}} = \frac{1}{4}yr$

II. In this case, the lifetime H of the system is $H = \max(\tau_1, \tau_2)$. The c.d.f. of H is, for $t \geq 0$,

$$\begin{aligned} F_H(t) &= \mathbb{P}(H \leq t) = \mathbb{P}(\max(\tau_1, \tau_2) \leq t) \\ &= \mathbb{P}(\tau_1 \leq t, \tau_2 \leq t) = \mathbb{P}(tau_1 \leq t)\,\mathbb{P}(\tau_2 \leq t) \\ &= F_{\tau_1}(t) F_{\tau_2}(t) = (1 - e^{-\lambda t})^2 \end{aligned}$$

Therefore, H is not an Exp RV, So in this case the process is not Poisson. THe p.d.f. of H is

$$f_H(t) = F'_H(t) = \begin{cases} 0 & t < 0. \\ 2(1 - e^{-\lambda t})e^{-\lambda t}\lambda & t > 0. \end{cases}$$

The average time elapsed between two consetutive failure is

$$\begin{aligned} \mathbb{E}[H] &= \int_{\mathbb{R}} t f_H(t) dt = \int_0^\infty t * 2(1 - e^{-\lambda t})e^{-\lambda t}\lambda dt \\ &= \frac{3}{4}yr \end{aligned}$$

III. In this case, the lifetime W of the system is equal to the sum of the lifetime of the two components: $W = \tau_1 + \tau_2$, so that $W \sim \Gamma(2, \lambda)$. The simplest way to find $\mathbb{E}[W]$ is $\mathbb{E}[W] = \mathbb{E}[\tau_1] + \mathbb{E}[\tau_2] = \frac{1}{\lambda} + \frac{1}{\lambda} = 1yr$

Problem 3

The toll collected from the traffic passing through the toll booth on Highway 44 between Oklahoma City and Tulsa can be modeled for the hours between 9 a.m. and 5 p.m. by a compound Poisson process. Assume that the toll booth serves the arriving vehicles instantaneously, so that there are no waiting lines.

The vehicles can be divided into two big categories personal vehicles and commercial vehicles. The personal vehicles arrive at the toll booth with average frequency 7 personal vehicles per minute, while the commercial vehicles arrive with average frequency 3 commercial vehicles per minute.

There are three types of personal vehicles 80% of the personal vehicles are cars, 15% are SUVs and 5% are RVs.

There are four types of commercial vehicles pick-up trucks, normal-size trucks, 18-wheelers, and busses; the probability with which a commercial vehicle belongs to each of these four types is 40%, 30%, 20%, and 10%, respectively.

The toll rates are the following: car \$1, SUV \$3, RV \$5; pick-up truck \$3, normal-size truck \$5, 18-wheeler \$8, bus \$10.

Please answer the questions below. Define clearly your notations, and use the concrete numbers given in this problem. You are allowed to use the theoretical results derived in class, but please write explicitly what results you use.

(a) Think of the toll collected from the personal vehicles as a compound Poisson process $Y_1(t)$, where at each arrival of a personal vehicle the collected toll is random. What is the p.m.f. of the random variable describing the collected toll from a personal vehicle? What is the rate of the Poisson process describing the moments of arrival of personal vehicles? the probability distribution of the inter-event times τ_j.

Ans:

$$Y_1(t) = \sum_{j=1}^{N_1(t)} X_{1,j}$$

The p.m.f of $X_{1,j}$ is

$$p_{X_1}(k) = \begin{cases} 0.8 & k = 1. \\ 0.15 & k = 3. \\ 0.05 & k = 5. \end{cases}$$

The rate of the Poisson Process $N_1(t)$ is $\lambda_1 = 7min^{-1}$

(b) Find the moment generating function of the process $Y_1(t)$, and use your result to find the average value of the toll collected in a period of 1 hour, and the variance of this toll.

Ans:

$$
\begin{aligned}
M_{Y_1(t)}(s) &= \mathbb{E}[e^{sY_1(t)}] = e^{\lambda_1 t[M_{X_1}(s)-1]} \\
M_{X_1}(s) &= \mathbb{E}[e^{sX_1}] = \sum_{i=1}^{3} e^{sx_i} p_{X_1}(x_i) \\
&= 0.8e^s + 0.15e^{3s} + 0.05e^{5s} \\
\Rightarrow M_{Y_1(t)}(s) &= e^{\lambda_1 t(0.8e^s+0.15e^{3s}+0.05e^{5s}-1)} \\
\mathbb{E}[Y_1(1hour)] &= \mathbb{E}[Y_1(60mins)] = \lambda_1 * 60 * \mathbb{E}[X_1] \\
\mathbb{E}[X_1] &= 1*0.8 + 3*0.15 + 5*0.05 = 1.5 \\
\Rightarrow \mathbb{E}[Y_1(1hour)] &= 7*60*1.5 = \$630 \\
Var[Y_1(1hour)] &= \lambda_1 * 60 * \mathbb{E}[X_1^2] = 1428
\end{aligned}
$$

(c) Answer the same questions as in part (a) about the Poisson process $Y_2(t)$ describing the toll collected from the commercial vehicles.

Ans:

$$Y_2(t) = \sum_{j=1}^{N_2(t)} X_{2,j}$$

The p.m.f of $X_{1,j}$ is

$$p_{X_2}(k) = \begin{cases} 0.4 & k = 3. \\ 0.3 & k = 5. \\ 0.2 & k = 8. \\ 0.1 & k = 10. \end{cases}$$

$\lambda_2 = 3min^{-1}$

(d) Answer the same questions as in (b), but for the process $Y_2(t)$.

Ans:

$$
\begin{aligned}
\mathbb{E}[X_2] = 5.30 &\Rightarrow \mathbb{E}[Y_2(1hour)] = \$954 \\
\mathbb{E}[X_2^2] = 33.9 &\Rightarrow Var[Y_2(1hour)] = 6102
\end{aligned}
$$

(e) Define the random process $Y(t) = Y_1(t) + Y_2(t)$ of the toll collected from all vehicles passing through the toll booth. We can think of this random process as a compound Poisson process. What is the frequency of the Poisson process with which the events of this random process occur? What is the p.m.f. of the toll collected from each vehicle passing through the toll booth (without making a distinction between personal and commercial vehicles)?

Ans:

$$
\begin{aligned}
Y_1(t) &= \sum_{j=1}^{N(t)} \tilde{X}_j \\
N(t) &= N_1(t) + N_2(t)
\end{aligned}
$$

The p.m.f of $\tilde{X}_j$ is a linear combination of $p_{X_1}(k)$ with weight $\frac{\lambda_1}{\lambda_1+\lambda_2} = 0.7$, and $p_{X_2}(k)$ with weight $\frac{\lambda_2}{\lambda_1+\lambda_2} = 0.3$

$$p_{\tilde{X}}(k) = \begin{cases} 0.7 * 0.8 = 0.56 & k = 1. \\ 0.7 * 0.15 + 0.3 * 0.4 = 0.225 & k = 3. \\ 0.7 * 0.05 + 0.3 * 0.3 = 0.125 & k = 5. \\ 0.3 * 0.2 = 0.06 & k = 8. \\ 0.3 * 0.1 = 0.03 & k = 10. \end{cases}$$

(f) Write explicitly the moment generating function of the random process $Y(t)$, as well as $\mathbb{E}[Y(t)]$ and the variance of $Y(t)$. On average, how much toll will be collected from 10 a.m. to 11 a.m.?

Ans:

$$\begin{aligned} M_{Y(t)}(s) &= \mathbb{E}[e^{sY(t)}] = e^{\lambda t[M_{\tilde{X}}(s)-1]} \text{ where } \lambda = 10min^{-1} \\ M_{\tilde{X}}(s) &= 0.56e^{s} + 0.225e^{3s} + 0.125e^{5s} + 0.06e^{8s} + 0.03e^{10s} \\ \mathbb{E}[\tilde{X}] &= 2.64 \Rightarrow \mathbb{E}[Y(1hour)] = \$1584 \\ \mathbb{E}[X_2^2] &= 12.55 \Rightarrow Var[Y_2(1hour)] = 7530 \end{aligned}$$

Name:Chao Jiang
MATH 5763 — Fall 2015 **HW 7** Due: 10/30/2015

Food for Thought Problem 1[1]

A sequence $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ of real numbers can contain a lot of information. One concise way of storing this information is to wrap up the numbers a_n together in a generating function. For example, let us define the (ordinary) *generating function* of the sequence **a** in the function $\mathbf{G}_a$ defined by

$$G_{\mathbf{a}}(s) = \sum_{n=0}^{\infty} a_n S^n \quad \text{for those } s \in \mathbb{R} \text{ for which the sum converges}$$

The sequence **a** may be reconstructed from the function $\mathbf{G}_a$ by setting $a_n = \frac{1}{n!}G_{\mathbf{a}}^{(n)}(0)$, where $f^{(n)}$ denotes the nth derivative of the function f. Generating functions are considered in many books on combinatorics, discrete mathematics, and probability (among others); a readable and freely available book is Herbert Wilfs *generatingfunctionology* (its second edition is freely available at https://www.math.upenn.edu/ wilf/DownldGF.html.

The *convolution* of the sequences $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ and $\mathbf{b} = \{b_n\}_{n=0}^{\infty}$is the sequence $\mathbf{c} = \{c_n\}_{n=0}^{\infty}$ defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \left(= \sum_{k=0}^{n} a_n - k b_n \right)$$

Sometimes the convolution of **a** and **b** is denoted by $\mathbf{a} * \mathbf{b}$.

(a) Let $z_n = (\cos\theta + i \sin\theta)^n$, where $i = \sqrt{-1}$, and θ is a fixed real number. Show that the generating function of the sequence $\mathbf{z} = \{z_n\}_{n=0}^{\infty}$ is

$$G_{\mathbf{z}}(s) = \frac{1}{1 - s(\cos\theta + i\sin\theta)} \qquad \text{for } |s| < 1$$

(b) Prove that, if **a** and **b** have generating functions $G_{\mathbf{a}}$ and $G_{\mathbf{b}}$, then the generating function of $\mathbf{c} = \mathbf{a} * \mathbf{b}$ is $G_{\mathbf{c}}(s) = G_{\mathbf{a}}(s)G_{\mathbf{b}}(s)$.

(c) Obtain the combinatorial identity $\sum_{k=0}^{N} \binom{N}{n}^2 = \binom{2N}{N}$, where $N \in \mathbb{N}$, by noticing that its left-hand side can be thought of as the convolution of the sequence

$$a_n = \begin{cases} \binom{N}{n}, & \text{for } 0 \leq n \leq N, \\ 0, & \text{for } n \geq N+1, \end{cases}$$

with itself, and using the fact proved in part (b) about $G_{\mathbf{a}*\mathbf{a}}$.

(d) It is a well-known fact from elementary probability (and an easy exercise for you) that if X and Y are independent random variables taking values in $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$, then the probability mass function of their sum, $Z = X + Y$, is the convolution of the probability mass functions of X and Y : $p_Z = p_X p_Y$. Here we think of a p.m.f. p_X as a sequence **a**

[1]Foot for Thought problems are for you to think about, but they do not need to be turned in with the regular homework.

with $a_n = p_X(n) = \mathbb{P}(X = n)$.
Let X and Y be independent Poisson random variables with parameters λ and μ, respectively. Compute explicitly the probability generating functions

$$G_X(s) = \sum_{n=0}^{\infty} p_X(n)s^n \quad \text{and} \quad G_Y(s) = \sum_{n=0}^{\infty} p_Y(n)s^n$$

and use them to show that $X + Y$ is a Poisson random variable and to find the parameter of the distribution of $X + Y$.

Food for Thought Problem 2

Let the function $F : \mathbb{R} \to [0, 1]$ shown in Figure 1 be defined by $F(x) = 0$ for $x \leq 0$, $F(x) = 1$ for $x \geq 1$, and

$$F(x) = \frac{1}{2^{j-1}} \quad \text{for} \quad \frac{1}{2^j} \leq x < \frac{1}{2^{j-1}},\ j \in \{1, 2, 3, \ldots\}$$

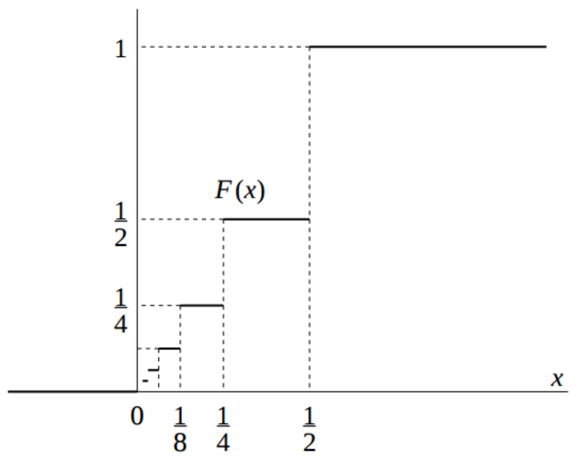

Figure 1: Graph of the function F.

Show that $\int_{\mathbb{R}} dF(x) = 1$, $\int_{\mathbb{R}} x dF(x) = \frac{1}{3}$, $\int_{\mathbb{R}} \ln x dF(x) = -2\ln 2$
*Hint:*You will need that, for $|q| < 1$, $\sum_{k=1}^{\infty} kq^{k-1} = \frac{d}{dq}\sum_{k=0}^{\infty} q^k = \frac{d}{dq}\frac{1}{1-q} = \frac{1}{(1-q)^2}$