Name:Chao Jiang

MATH 5763 — Fall 2015 **HW 6** Due: 10/21/2015

Problem 1

Let $N = \{N_t : t \geq 0\}$ be a (time-homogeneous) Poisson process with rate λ. Define the flip-flop process $X = X_t : t \geq 0$ with state space $S = \{0, 1\}$ by

$$X_t = \frac{1}{2} + (-1)^{N_t}\left[X_0 - \frac{1}{2}\right],$$

where X_0 is a random variable with values in S; assume that X_0 is independent of the Poisson process N. In other words, the flip-flop process switches between the states 0 and 1 at each event of N. Since N is a Markov chain, X is also a Markov chain. Let $\mathbf{P_t} = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix}$ be the stochastic semigroup of the process X, and $\mathbf{G}$ be the generator of $\mathbf{P_t}$.

(a) Find the short-time transition probabilities $p_{ij}(h) = \mathbb{P}(X_{t+h} = j | X_t = i)$ and show that the generator of the stochastic process X is $\mathbf{G} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$

Ans:

$$\begin{aligned} p_{ij}(h) &= \mathbb{P}(X_{t+h} = j | X_t = i) \\ &= \begin{cases} \lambda h + o(h), & \text{for } j = i + 1, \\ 1 - \lambda h + o(h), & \text{for } j = i, \\ o(h), & \text{for otherwise,} \end{cases} \end{aligned}$$

So

$$\mathbf{P_h} = \begin{pmatrix} 1 - \lambda h + o(h) & \lambda h + o(h) \\ \lambda h + o(h) & 1 - \lambda h + o(h) \end{pmatrix}$$

then

$$\mathbf{G} = \frac{d\mathbf{P_h}}{dh} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$$

(b) To find the time evolution of the chain X i.e., to find the stochastic semigroup $\mathbf{P}_t = e^{t\mathbf{G}}$, one needs to find $\mathbf{G}^n$ for $n \in \mathbb{N}$ (of course, $G_0 = \mathbb{I}$, the identity matrix).

One way to compute $\mathbf{G}^n$ is to diagonalize it by a similarity transformation, $\tilde{\mathbf{G}} = \mathbf{M}^{-1}GM$, using the tricks learned in Lecture 7. Then compute the nth power of the diagonal matrix $\tilde{\mathbf{G}}$ (which is very easy), and finally use that $\tilde{\mathbf{G}}^n = \mathbf{M}^{-1}G^n\mathbf{M}$, so that $\mathbf{G}^n = \mathbf{M}\tilde{\mathbf{G}}^n\mathbf{M}^{-1}$. In fact, one can directly compute the diagonal matrix $e^{t\mathbf{G}}$, and then to use that

$$\mathbf{P}_t = e^{t\mathbf{G}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{G}^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{M}\tilde{\mathbf{G}}^n\mathbf{M}^{-1} = \mathbf{M}\sum_{n=0}^{\infty} \frac{t^n}{n!}\tilde{\mathbf{G}}^n\mathbf{M}^{-1} = \mathbf{M}e^{t\mathbf{G}}\mathbf{M}^{-1}$$

You may use that in this problem one can take $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}. \end{pmatrix}$

At the end, you should obtain that $\mathbf{P}_t = \begin{pmatrix} \frac{1}{2}(1 + e^{-2\lambda t}) & \frac{1}{2}(1 - e^{-2\lambda t}) \\ \frac{1}{2}(1 - e^{-2\lambda t}) & \frac{1}{2}(1 + e^{-2\lambda t}) \end{pmatrix}$, but I would like to see the details of your computations.

Ans:

$$\tilde{\mathbf{G}} = \mathbf{M}^{-1}\mathbf{G}\mathbf{M} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2\lambda \end{pmatrix}$$
$$\mathbf{P}_t = \mathbf{M}e^{t\tilde{\mathbf{G}}}\mathbf{M}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\lambda t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+e^{-2\lambda t}) & \frac{1}{2}(1-e^{-2\lambda t}) \\ \frac{1}{2}(1-e^{-2\lambda t}) & \frac{1}{2}(1+e^{-2\lambda t}) \end{pmatrix}$$

(c) Now you will find $\mathbf{P}_t$ directly, without using the generator $\mathbf{G}$. (Of course, you have to pretend that you dont know the answer.) One can do this using several methods.

The standard method is to solve, say, the Kolmogorov backward equations, $\frac{d}{dt}\mathbf{P}_t = \mathbf{G}\mathbf{P}_t$, with appropriate initial conditions (the Kolmogorov forward equations can also be used). You do not need to do it here because we did this in class (see the example on pages 129131 of Lefevbres book, and pages 2124 of the lecture notes from Lecture 7).

A trickier method for computing $\mathbf{P}_t$ (which works in this particular problem) is the following. Note that $p_{01}(t) = \mathbb{P}(X_{s+t} = 1|X_s = 0)$ is equal to the probability that there were an odd number of events of the Poisson process N of intensity λ in the interval $(s, s+t]$. Using the explicit expression for the probability of exactly k events of a Poisson process to occur in a time interval of length t, compute $p_{01}(t)$.

From $p_{01}(t)$, one can easily find $p_{00}(t)$ (how?), and the values of $p_{10}(t)$, and $p_{11}(t)$ can be obtained simply by relabeling, but you do *not* need to do this here

Hint: Note that

$$\sum_{j\ odd} \frac{\alpha^j}{j!} = \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!} = \frac{1}{2}(e^{\alpha} - e^{-\alpha})$$

Ans:

$$p_{01}(t) = \sum_{k\ odd}^{\infty} e^{(-\lambda t)} \frac{(\lambda t)^k}{k!} = \frac{1}{2}(1 - e^{-2\lambda t})$$

Considering that $p_{00}(t) + p_{01}(t) = 1$, So $p_{00}(t) = \frac{1}{2}(1 + e^{-2\lambda t})$

(d) Find the stationary distribution π by using the generator $\mathbf{G}$.

Ans:

$$\pi\mathbf{G} = 0 \Rightarrow (\pi_0, \pi_1) \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$$
$$1 = \pi_0 + \pi_1$$

By solving the above equations, we obtian $\pi = (\frac{1}{2}, \frac{1}{2})$

(e) Now assume that initially the chain $\mathbf{X}$ is in state 0 (i.e., that $X_0 = 0$). Determine the probability distribution $\mathbf{p}(t) = (p_0(t)\ p_1(t))$ (where $p_j(t) = \mathbb{P}(X_t = j)$) of the chain X at time t by using your results above. As t goes to infinity, does $\mathbf{p}(t)$ tend to the stationary distribution π?

Ans:

$$p_0(t) = \lim_{t\to\infty} \frac{1}{2}(1 + e^{-2\lambda t}) = \frac{1}{2}$$
$$p_1(t) = \lim_{t\to\infty} \frac{1}{2}(1 - e^{-2\lambda t}) = \frac{1}{2}$$

We known that $\mathbf{p}(t)$ tend to the stationary distribution π.

(f) Define the generating functions $G_i(\xi, t) := \sum_{j=0}^{1} p_{ij}(t)\xi^j$, and show that G_i satisfies the firstorder partial differential equation $\frac{\partial G_i}{\partial t} + 2\lambda(\xi - 1)\frac{\partial G_i}{\partial \xi} = \lambda(\xi - 1)$. Since the proofs for G_0 and G_1 are essentially the same, give a proof only for G_0. You have to do this by using the Kolmogorov forward equations, $\frac{d}{dt}\mathbf{P}_t = \mathbf{P}_t\mathbf{G}$, i.e.,

$$\begin{pmatrix} p'_{00}(t) & p'_{01}(t) \\ p'_{10}(t) & p'_{11}(t) \end{pmatrix} = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix} \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$$

Ans: We known that:

$$\begin{aligned}
p'_{00}(t) &= -\lambda p_{00}(t) + \lambda p_{01}(t) \\
p'_{01}(t) &= \lambda p_{00}(t) - \lambda p_{01}(t) \\
G_0 &= \sum_{j=0}^{1} p_{0j}(t)\xi^j = p_{00}(t) + p_{01}(t)\xi \\
\frac{\partial G_0}{\partial t} &= p'_{00}(t) + p'_{01}(t)\xi \\
\frac{\partial G_0}{\partial \xi} &= p_{01}(t)
\end{aligned}$$

Substituting the above equations into the left of the equation is

$$\begin{aligned}
\frac{\partial G_i}{\partial t} + 2\lambda(\xi - 1)\frac{\partial G_i}{\partial \xi} &= p'_{00}(t) + p'_{01}(t)\xi + 2\lambda(\xi - 1)p_{01}(t) \\
&= -\lambda p_{00}(t) + \lambda p_{01}(t) + (\lambda p_{00}(t) - \lambda p_{01}(t))\xi + 2\lambda(\xi - 1)p_{01}(t) \\
&= \lambda(\xi - 1)p_{01}(t) + \lambda(\xi - 1)p_{00}(t) \\
&= \lambda(\xi - 1)(p_{00}(t) + p_{01}(t)) \Longleftarrow p_{00}(t) + p_{01}(t) = 1 \\
&= \lambda(\xi - 1)
\end{aligned}$$

(g) What are the initial conditions that G_0 and G_1 must satisfy? Why? (Recall that we assumed that initially the chain is in state 0.)

One can show that the solution of the PDE above and the corresponding initial conditions is

$$G_0(\xi, t) = \frac{1}{2}\left[1 + \xi + (1 - \xi)e^{-2\lambda t}\right], G_1(\xi, t) = \frac{1}{2}\left[1 + \xi - (1 - \xi)e^{-2\lambda t}\right]$$

Ans:

$$G_0(\xi,0) = \frac{1}{2}\left[1+\xi+(1-\xi)e^0\right] = 1$$
$$G_1(\xi,0) = \frac{1}{2}\left[1+\xi-(1-\xi)e^0\right] = \xi$$

(h) From the very definition of $G_i(\xi,t)$, show that

$$\mathbb{E}[X_t|X_0=i] = \frac{\partial G_i}{\partial \xi}(1,t)$$

and

$$Var[X_t|X_0=i] = \mathbb{E}[X_t^2|X_0=i] - \mathbb{E}[X_t|X_0=i]^2 = \frac{\partial^2 G_i}{\partial^2 \xi}(1,t) + \frac{\partial G_i}{\partial \xi}(1,t) - \left[\frac{\partial G_i}{\partial \xi}(1,t)\right]^2$$

Hint: Compare with Food for Thought Problem 1 below

Ans: For the first equation, we have

$$\mathbb{E}[X_t|X_0=i] = \sum_{X_0} X_t p[X_t|X_0] = p_{01}(t) + p_{11}(t)$$
$$\frac{\partial G_i}{\partial \xi}(1,t) = p_{01}(t) + p_{11}(t)$$

For the last equation,

$$Var[X_t|X_0=i] = \mathbb{E}[X_t^2|X_0=i] - \mathbb{E}[X_t|X_0=i]^2$$
$$= p_{01}(t) + p_{11}(t) - \Big(p_{01}(t) + p_{11}(t)\Big)^2$$
$$\frac{\partial^2 G_i}{\partial^2 \xi}(1,t) + \frac{\partial G_i}{\partial \xi}(1,t) - \left[\frac{\partial G_i}{\partial \xi}(1,t)\right]^2 = p_{01}(t) + p_{11}(t) - \Big(p_{01}(t) + p_{11}(t)\Big)^2$$

(i) Use the concrete expressions for the generating functions $G_i(\xi,t)$ of the flip-flop problem (written in part (g)) in order to find the conditional expectation $\mathbb{E}[X_t|X_0=0]$ and the conditional variance $Var[X_t|X_0=0]$, and sketch $\mathbb{E}[X_t|X_0=0]$ and $Var[X_t|X_0=0]$ as functions of t. Do your results look reasonable in the limiting cases $t \to 0^+$ and $t \to \infty$? Explain briefly.

Ans:

$$\mathbb{E}[X_t|X_0=0] = p_{01}(t)$$
$$Var[X_t|X_0=0] = p_{01}(t) - (p_{01}(t))^2$$
$$\lim_{t\to 0^+} \mathbb{E}[X_t|X_0=0] = 0$$
$$\lim_{t\to 0^+} Var[X_t|X_0=0] = 0.25$$
$$\lim_{t\to \infty} \mathbb{E}[X_t|X_0=0] = 0.5$$
$$\lim_{t\to \infty} Var[X_t|X_0=0] = 0.5$$

Name:Chao Jiang
MATH 5763 — Fall 2015 **HW 6** Due: 10/21/2015

Problem 2

A *death process* is a random process that describes the number of people in a society where the only reason for changing the number of people is dying (nobody is born, there is no immigration, etc.). We say that the random process $X = X_t : t \geq 0$ is a death process with parameter μ if each person dies independently of every other person, and the probability that each person dies in one unit of time is μ (we assume that the units of time we use are much shorter than the average lifetime of the people). Clearly, the probability of a death of a person in one unit of time in a population of i people is $i\mu$ (again, we assume that the unit of time is short). Here is the precise mathematical definition of a death process with parameter μ:

- the state space of the process is $Z_+ = \{0, 1, 2, 3, ...\}$;
- the process is non-increasing, i.e., if $s < t$, then $X_s \geq X_t$;
- if h is a very small positive number, then, for $j \in N = \{1, 2, 3, ...\}$,

$$\mathbb{P}(X_{t+h} = j | X_t = i) = \begin{cases} i\mu h + o(h) & \text{if } j = i - 1, \\ 1 - i\mu h + o(h) & \text{if } j = i. \\ o(h) & \text{if } j > i \text{ or } j \leq i - 2. \end{cases}$$

and

$$\mathbb{P}(X_{t+h} = 0 | X_t = i) = \begin{cases} \mu h + o(h) & \text{if } i = 1, \\ 1 & \text{if } i = 0. \\ o(h) & \text{if } j > i. \end{cases}$$

Here $o(h)$ is a function satisfying $\lim_{h \to 0} \frac{o(h)}{h} = 0$

- for $s < t$, the difference $X_t X_s$ (equal to the number of deaths in the interval $(s, t]$) does not depend on what has happened in the time interval $(0, s]$.

In this problem you will analyze some aspects of this process.

(a) Let $p_i(t) = \mathbb{P}(X_t = i)$. Condition on X_t to derive the equations

$$\begin{aligned} p_0(t+h) &= \mu h p_1(t) + p_0(t) + o(h), \\ p_j(t+h) &= (j+1)\mu h p_{j+1}(t) + (1 - j\mu h)p_j(t) + o(h), j \in \mathbb{N} \end{aligned}$$

Ans:

$$\begin{aligned} p_0(t+h) &= \mathbb{P}(X_{t+h} = 0) = \mathbb{P}(X_{t+h} = 0 | X_t = 1)\,\mathbb{P}(X_t = 1) + \mathbb{P}(X_{t+h} = 0 | X_t = 0)\,\mathbb{P}(X_t = 0) \\ &= (\mu h + o(h))\,\mathbb{P}(X_t = 1) + 1 * \mathbb{P}(X_t = 0) \\ &= \mu h p_1(t) + p_0(t) + o(h) \\ p_j(t+h) &= \mathbb{P}(X_{t+h} = j | X_t = j+1)\,\mathbb{P}(X_t = j+1) + \mathbb{P}(X_{t+h} = j | X_t = j)\,\mathbb{P}(X_t = j) \\ &= (j+1)\mu h p_{j+1}(t) + (1 - j\mu h)p_j(t) + o(h), j \in \mathbb{N} \end{aligned}$$

(b) Subtract $p_j(t)$ from the jth equation from (a), divide through by h, and take the limit $h \to 0$, to obtain the system

$$\begin{aligned} p_0'(t) &= \mu p_1(t), \\ p_j'(t) &= (j+1)\mu p_{j+1}(t) - j\mu p_j(t), j \in \mathbb{N} \end{aligned}$$

Let the initial condition be $X_0 = I$, where I is a random variable taking values in $\mathbb{Z}_+$.

Ans:

$$\begin{aligned} p_0'(t) &= \lim_{h\to 0} \frac{p_0(t+h) - p_0(t)}{h} = \mu p_1(t) \\ p_j'(t) &= \lim_{h\to 0} \frac{(j+1)\mu h p_{j+1}(t) + (1 - j\mu h)p_j(t) + o(h) - pj(t)}{h} \\ &= (j+1)\mu p_{j+1}(t) - j\mu p_j(t), j \in \mathbb{N} \end{aligned}$$

(c) Define the *generating function*

$$\Delta(\xi, t) := \sum_{j=0}^{\infty} p_j(t)\xi^j$$

ans how that

$$\frac{\partial \Delta}{\partial \xi} = \sum_{j=0}^{\infty} j p_j(t)\xi^{j-1} = \sum_{j=1}^{\infty} j p_j(t)\xi^{j-1}, \frac{\partial \Delta}{\partial t} = \sum_{j=0}^{\infty} p_j'(t)\xi^j$$

Ans:

$$\begin{aligned} \frac{\partial \Delta}{\partial \xi} &= \sum_{j=0}^{\infty} j p_j(t)\xi^{j-1} = \sum_{j=1}^{\infty} j p_j(t)\xi^{j-1} \Longleftarrow j = 0 \text{ can be omit} \\ \frac{\partial \Delta}{\partial t} &= \sum_{j=0}^{\infty} p_j'(t)\xi^j \end{aligned}$$

(d) Use the differential equations from part (b) to show that $\Delta(\xi, t)$ of the death process satisfies the partial differential equation

$$\frac{\partial \Delta}{\partial t} = \mu(1 - \xi)\frac{\partial \Delta}{\partial \xi}$$

and the initial condition $\Delta(\xi, 0) = \xi^I$ (where $I = X_0$ is the initial population).

Hint: Multiply the differential equation for $p_j'(t)$ by ξ^j and add all the equations.

Ans:

$$\begin{aligned}\frac{\partial\Delta}{\partial t} &= \sum_{j=0}^{\infty} p_j'(t)\xi^j = \sum_{j=0}^{\infty}(j+1)\mu p_{j+1}(t) - j\mu p_j(t)\xi^j \\ &= \mu\sum_{j=0}^{\infty}(j+1)p_{j+1}(t)\xi^j - \mu\sum_{j=0}^{\infty} jp_j(t)\xi^j \\ \mu(1-\xi)\frac{\partial\Delta}{\partial\xi} &= \mu(1-\xi)\sum_{j=1}^{\infty} jp_j(t)\xi^{j-1} \\ &= \mu\sum_{j=1}^{\infty} jp_j(t)\xi^{j-1} - \mu\sum_{j=1}^{\infty} jp_j(t)\xi^j\end{aligned}$$

Actually

$$\begin{aligned}\mu\sum_{j=0}^{\infty}(j+1)p_{j+1}(t)\xi^j - \mu\sum_{j=0}^{\infty} jp_j(t)\xi^j &= \mu\sum_{k=1}^{\infty} kp_k(t)\xi^{k-1} - \mu\sum_{j=1}^{\infty} jp_j(t)\xi^j \\ &= \mu\sum_{j=1}^{\infty} jp_j(t)\xi^{j-1} - \mu\sum_{j=1}^{\infty} jp_j(t)\xi^j\end{aligned}$$

(e) How can the probabilities $p_j(t) = \mathbb{P}(X_t = j)$ be expressed in terms of $\xi-$ derivatives of $\Delta(\xi, t)$ evaluated at $\xi = 0$? Use this to find the explicit expressions for $\mathbb{P}(X_t = 0)$ and $\mathbb{P}(X_t = 1)$, using that the solution of the initial-value problem for the generating function posed in part (d) is

$$\Delta(\xi, t) = [1 + (\xi - 1)e^{-\mu t}]^I$$

(there is no need to derive this expression).

Ans:

$$\begin{aligned}\mathbb{P}(X_t = 0) &= \Delta(0, t) = [1 - e^{-\mu t}]^I \\ \mathbb{P}(X_t = 1) &= \Delta'(0, t) = I[1 - e^{-\mu t}]^{I-1}e^{-\mu t} \Longleftarrow \text{with r.s.t } \xi\end{aligned}$$

(f) Show that

$$\frac{\partial\Delta}{\partial\xi}(1, t) = \mathbb{E}[X_t],$$

and use this fact to find $\mathbb{E}[X_t]$ for the death process.

Ans:

$$\begin{aligned}\mathbb{E}[X_t] &= \sum_{j=0}^{\infty} jp_j(X_t = j) \\ \frac{\partial\Delta}{\partial\xi}(1, t) &= \sum_{j=0}^{\infty} jp_j(t)\xi^{j-1} = \sum_{j=0}^{\infty} jp_j(X_t = j)\end{aligned}$$

So

$$\frac{\partial \Delta}{\partial \xi}(1,t) = \mathbb{E}[X_t] = I[1 + (\xi - 1)e^{-\mu t}]^{I-1}e^{-\mu t}|_{\xi=1} = Ie^{-\mu t},$$

(g) Using the same idea as in part (f), express the variance of X_t in terms of derivatives of its generating function evaluated at $\xi = 1$. Use the explicit expression for $\Delta(\xi, t)$ given in (e) to find Var X_t for the death process.

Ans:

$$\begin{aligned}
\mathrm{Var} X_t &= \Delta''(1,t) + \Delta'(1,t) - (\Delta'(1,t))^2 \\
&= I(I-1)e^{-2\mu t} + Ie^{-\mu t} - I^2 e^{-2\mu t} \\
&= I(e^{-\mu t} - e^{-2\mu t})
\end{aligned}$$

(h) Radioactive decay is an example of a death process, if we think of a nucleus of the radioactive isotope as alive before it decays, and dead after that. The half-life, $T_{1/2}$, of a radioactive isotope is defined as the time after which only half of the initial number of nuclei of this isotope are alive. How is $T_{1/2}$ related to μ? Justify your claim.

Ans: Suppose there are I in the begining, then after $T_{1/2}$ time, only $\frac{I}{2}$. The probability that $T_{1/2} = h$ is $\mathbb{P} = \mathbb{P}(X_{t+h} = 0|X_t = 1)^{2/I} = (\mu h + o(h)^{2/I})$

Name:Chao Jiang
MATH 5763 — Fall 2015 **HW 6** Due: 10/21/2015

Food for Thought Problem 1[1]

The *probability generating function* (p.g.f.) of a random variable J taking only values in $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$ is defined as

$$G_j(\xi) := \mathbb{E}[\xi^j] = \sum_{j=0}^{\infty} p_j \xi^j, \quad \text{where } p_j = \mathbb{P}(J = j).$$

provided the right-hand side exists. Prove the following properties of the p.g.f.s:

(a) $G_J(1) = 1$;

(b) $G_j'(1) = \mathbb{E}[J]$;

(c) $G_j''(1) = \mathbb{E}[J^2] - \mathbb{E}[J]$;

(d) Var $J = G''_J(1) + G'_J(1) - [G_j'(1)]^2$

(e) if $J_1, J_2, \cdots, J_r$ are i.i.d. random variables taking values in $\mathbb{Z}_+$, and $K = J_1 + \cdots + Jr$, then

$$G_K(x) = [G_j(x)]^r$$

Please write explicitly where you use each of the assumptions.

[1] Foot for Thought problems are for you to think about, but they do not need to be turned in with the regular homework.