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MATH 5763 — Fall 2015 **HW 5** Due: 10/06/2015

Problem 1

This problem is a continuation of Problems 3 and 4 of Homework 4. In those problems you considered a Markov chain with five states, and after relabeling the states the transition matrix of the Markov chain became

$$\mathbf{P} = \left(\begin{array}{cc|c} \mathbf{C_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C_2} & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array}\right) = \left(\begin{array}{cc|c|cc} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{array}\right);$$

here $\mathbf{0}$ denotes a matrix of appropriate size with zero entries, while a star denotes an arbitrary matrix of appropriate size. You proved that $\mathbf{C_1}$ and $\mathbf{C_2}$ are stochastic matrices, while $\mathbf{T}$ is not. The form in which we wrote the transition probabilities $\mathbf{P}$ (after relabeling the states) is very convenient for studying the long-term behavior of the Markov chain because of the following fact:

$$\mathbf{P}^n = \left(\begin{array}{cc|c} \mathbf{C_1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C_2} & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array}\right)^n = \left(\begin{array}{cc|c} \mathbf{C_1^n} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C_2^n} & \mathbf{0} \\ \hline * & * & \mathbf{T^n} \end{array}\right).$$

(a) **[Food for Thought (discussed in class)]** Consider only the irreducible matrix $\mathbf{C_1}$ containing only the recurrent states 1 and 2. Let μ_i be the average number of transitions needed by the process, starting from state i, to return to i for the first time (sometimes μ_i is called the *mean recurrence time* of state i). In part (a) of Problem 4 of Homework 4 you found the probabilities $\rho_{ii}^{(n)}$ of returning to state i from the initial state i for the first time in exactly n steps (for $i \in \{1, 2\}$). Use your results to compute μ_1 and μ_2. Are the states 1 and 2 positive recurrent or null recurrent? Did you expect what you just observed? Explain briefly. *Hint*: The following trick is very useful for evaluating sums: differentiating with respect to q both sides of the formula for the sum of a geometric series,

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \quad |q| < 1,$$

one obtains

$$\sum_{n=0}^{\infty} nq^{n-1} = \frac{1}{(1-q)^2}, \quad |q| < 1,$$

(in the sum in the left-hand side one can start the summation from 0, but the term with $n = 0$ is equal to zero). (Incidentally, differentiating one more time, one can obtain an expression for $\sum_{n=1}^{\infty} n(n-1)q^n$, from which $\sum_{n=2}^{\infty} n^2 q^n$ can be found, etc.)

Ans:

$$\mu_1 = \sum_{n=1}^{\infty} n\rho_{1,1}^{(n)} = \frac{5}{3}$$
$$\mu_2 = \sum_{n=1}^{\infty} n\rho_{2,2}^{(n)} = \sum_{n=1}^{\infty} 2n(\frac{1}{3})^{(n-1)} - 2 = 2 * \frac{1}{(1-\frac{1}{3})^2} - 2 = \frac{5}{2}$$

(b) [**Food for Thought (discussed in class)**] Consider only the matrix $\mathbf{C_1}$ containing the recurrent states 1 and 2. Since these two states form an irreducible set, the Ergodic Theorem guarantees that it has a unique stationary distribution $\tilde{\pi} = (\tilde{\pi}_\mathbf{1}, \tilde{\pi}_\mathbf{2})$. Find $\tilde{\pi}$.

Ans:

$$\pi = \pi\mathbf{P}, \text{where } \pi = (\pi_1 \ \pi_2)$$
$$1 = \sum_{i=1}^{2} \pi_i$$

By solving the equations above, we got $\pi = (\frac{3}{5}, \frac{2}{5})$

(c) [**Food for Thought (discussed in class)**] Your results in (a) and (b) are related. How?

Ans: $\tilde{\pi} = (\frac{1}{\mu_1}, \frac{1}{\mu_2}) = (\frac{3}{5}, \frac{2}{5})$.

(d) In the rest of this problem you will consider the transient states 4 and 5. Recall that the indicator function of an event $A \subseteq \Omega$ is a random variable $I_A : \Omega \to \{0, 1\}$ defined as

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \in A. \end{cases}$$

Let j be a transient state,

$$Y_j := \sum_{n=0}^{\infty} I_{\{X_n = j\}}$$

be the number of times the chain visits it, and

$$E[Y_j | X_0 = k] = E\left[\sum_{n=0}^{\infty} I_{\{X_n=j\}} \middle| X_0 = k \right]$$

be the expected number of times the chain visits the transient state j if initially it is in the transient state k. Show that

$$E[Y_j | X_0 = k] = \sum_{n=0}^{\infty} p_{kj}^{(n)} = \left(\sum_{n=0}^{\infty} \mathbf{P}^{(n)} \right)_{kj} = \left(\sum_{n=0}^{\infty} \mathbf{P}^n \right)_{kj} = \left((\mathbf{I} - \mathbf{T})^{-1} \right)_{kj}$$

where $\mathbf{I}$ is the unit matrix of appropriate size.

Hint: Use the fact that, for a matrix $\mathbf{A}$, if the geometric series $\sum_{n=0}^{\infty} \mathbf{A}^n$ converges, its sum is equal to

$$\sum_{n=0}^{\infty} \mathbf{A}^n = (\mathbf{I} - \mathbf{A})^{-1}$$

Note that for numbers (i.e., 1×1 matrices) this becomes the well-known formula.

Remark: If $\mathbf{T}$ corresponds to the transient states only, there is a theorem that guarantees that the matrix $\mathbf{I} - \mathbf{T}$ is invertible.

Ans:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

we know that each entry of $\mathbf{P}$ no more than 1. So this matrix is converges. also $\sum_{n=0}^{\infty} P^n = \sum_{n=0}^{\infty} T^n$, and the matrix $\mathbf{I} - \mathbf{T}$ is invertible. So $\left(\sum_{n=0}^{\infty} \mathbf{P}^n \right)_{kj} = \left((\mathbf{I} - \mathbf{T})^{-1} \right)_{kj}$

(e) I did the math, and obtained

$$\mathbf{T} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{I} - \mathbf{T} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \mathbf{I} - \mathbf{T}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}$$

Discuss the meaning of each entry of $(\mathbf{I} - \mathbf{T})^{-1}$ in the light of what you proved in part (d).

Ans: Each entry represents the expected number of times the chain visits the transient state j if initially in the stransient state k.

$$E[Y_4|X_0 = 4] = 2 \quad E[Y_5|X_0 = 4] = 1$$
$$E[Y_4|X_0 = 5] = 1 \quad E[Y_5|X_0 = 5] = \frac{2}{3}$$

Problem 2

In this and in the next problem you will consider a particular case of the gamblers ruin problem. It will be useful if you review the material from pages 8588 and 100104 of the book. In particular, you need to know the definitions of:

- the probability $\rho_{ij}^{(n)}$ of first visit of state j starting from state i in exactly n steps, where $n \geq 1$ (page 81);
- the probability $f_{ij} = \sum_{n=1}^{\infty} \rho_{ij}^{(n)}$ of eventually visiting state j starting from state i (page 87);
- the probability $r_i(C)$ of eventually entering the closed irreducible set C of recurrent states if initially the system is in the transient state i (page 100).

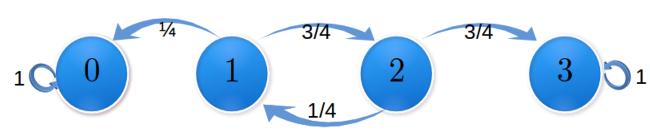
Name:Chao Jiang
MATH 5763 — Fall 2015 **HW 5** Due: 10/06/2015

Consider a four-state gamblers ruin chain with state space $\mathscr{X} = \{0, 1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Draw a picture with the states and arrows corresponding to non-zero transition probabilities p_{ij} between them. Identify the transient and the recurrent states, as well as the closed and irreducible classes of recurrent states.

Ans: The transient States are $\{1, 2\}$, the recurrent states are $\{0, 3\}$. See the picture below:



(b) Compute the probability $\rho_{10}^{(n)}$ for all $n \geq 1$.

Ans:

$$\rho_{10}^{(n)} = \begin{cases} (\frac{3}{4} * \frac{1}{4})^{(n-1)} * \frac{1}{4}, & \text{for } n \text{ is odd,} n \geq 1, \\ 0. & \text{for } n \text{ is even, } n \geq 1. \end{cases}$$

Hint: This is easy for the particular transition matrix in this problem! For example, it is clear that $\rho_{10}^{(n)} = 0$ for all even values of n.

(c) Use your result from (b) to show that $f_{10} = \frac{4}{13}$

$$f_{10} = \sum_{n=1}^{\infty} \rho_{10}^{(n)} \; n = 1, 3, 5, ...$$
$$= \sum_{n=0}^{\infty} \frac{1}{4} (\frac{3}{16})^{(n)} = \frac{1}{4} * \frac{1}{1 - \frac{3}{16}} = \frac{4}{13}$$

(d) Find the probability f_{13}. Explain briefly your reasoning.

Ans: $f_{13} = 1 - \frac{4}{13} = \frac{9}{13}$

Hint: If you think about the meaning of this probability, you wont need to do any calculations!

(e) Show that $f_{11} = \rho_{11}^{(2)}$ and use this to compute f_{11}.

Ans: From the above picture, we know that from state 1 to state 1, there is only one way, first move to state 2 and then move back to 1. no other ways considering that it cannot stay at state 2 and state 0 and 3 are closed and states. $f_{11} = \frac{3}{16}$

(f) Determine f_{22}.

Ans: $f_{22} = \frac{3}{16}$, the same reason as qeustion (e).

(g) Given that $X_0 = 1$, what is the expected number of time that the chain will return to state 1 in the subsequent steps by using the fact that, if the chain is at state i at time $n = 0$, the expected number of subsequent visits to a transient state j is equal to $\frac{f_{ij}}{1-f_{ij}}$ (You can take this fact for granted, but think about its meaning in the light of Equation (3.41) on page 88 of Lefevbres book.)

Ans: $\mathbb{E} = \frac{f_{11}}{1-f_{11}} = \frac{3}{13}$

(h) Answer the same question as in (g) but this time using the method you derived in Problem 1(d,e) above.

Ans: $\mathbf{T} = \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$, $\mathbb{E} = (\mathbf{I} - \mathbf{T}^{-1})_{11} - 1 = \frac{3}{13}$

Problem 3

In this problem you will find the probability of ruin of a gambler for the same transition probability matrix as in Problem 2, but using different methods.

Recall Theorem 3.2.2 of the book according to which, if C is a closed and irreducible set of recurrent states, and D is the set of all transient states, and $i \in D$ is a transient state, then the probability $r_i(C)$ of the chain to enter the set C eventually if $X_0 = i$ is the smallest nonnegative solution of the system

$$r_i(C) = \sum_{j \in D} p_{ij} r_j(C) + \sum_{j \in C} p_{ij}, \quad \text{for all } i \in D$$

Moreover, if D is a finite set (as in this problem), then the solution is unique.

(a) Write down this system of equations for the probabilities $r_i(0)$ of eventual ruin starting from state i.

Ans:

$$r_1(0) = \frac{3}{4} r_2(0) + \frac{1}{4}$$
$$r_2(0) = \frac{1}{4} r_1(0)$$

(b) Solve the system derived in (a).

Ans:$r_1(0) = \frac{4}{13}, r_2(0) = \frac{1}{13}$

(c) Compare your results with your results from Problem 2. Please be specific what numbers you are comparing.

Ans: It is the same as question (c) of Problem 2, f_{11}.

Name:Chao Jiang
MATH 5763 — Fall 2015 **HW 5** Due: 10/06/2015

Problem 4

Let W_t and M_t be the number of women, respectively men, entering a big store in the time interval $[0,t]$. Assume that $W = \{W_t\}_{t\geq 0}$ and $M = \{M_t\}_{t\geq 0}$ are independent Poisson processes with intensities ω and μ, respectively.

(a) **[Food for Thought only (follows from Problem 3(a) of HW 3; discussed in class)]** Prove that the total number of customers, $N_t := W_t + M_t$, entering the store forms a Poisson process with intensity $\omega+\mu$. Explain briefly why your result is obvious.

Ans: Possion process expressed in the form of exponential, considering these two process are independent, so the sum to these variable can also be expressed in the form of exponential with parameter $\omega + \mu$.

(b) **[Food for Thought only (follows from Problem 3(a) of HW 3; discussed in class)]** Find the conditional probability $\mathbb{P}(M_t = m|N_t = n)$, for $0 \leq m \leq n$, and show that M_t can be considered as a binomial random variable with one random and one deterministic parameter, namely $M_t \sim Bin(N_t, \frac{\mu}{\omega+\mu})$. Why is this result obvious? Show that the conditional expectation $\mathbb{E}[M_t|N_t]$ is the random variable $\mathbb{E}[M_t|N_t] = \frac{\mu}{\mu+\omega}N_t$. (To answer the last question, you can use what you know about binomial random variables.)

Ans:

$$
\begin{aligned}
P(M_t = m|N_t = n) &= \frac{P(M_t = m)P(W_t = n-m)}{P_{X+Y}(n)} \\
&= \frac{e^{-\mu}\frac{\mu^m}{m!}e^{-\omega}\frac{\omega^{n-m}}{(n-m)!}}{\frac{(\mu+\omega)^n}{n!}e^{-(\mu+\omega)}} \\
&= \frac{n!}{m!(n-m)!}\frac{\mu^m\omega^{n-m}}{(\mu+\omega)^n} \\
&= \binom{n}{m}(\frac{\mu}{(\mu+\omega)})^m(\frac{\omega}{\omega+\mu})^{n-m}
\end{aligned}
$$

So the distribution is binomial with parameters $(\frac{\mu}{\omega+\mu}, n)$. Considering the expectation of binomial is $\mathbb{E} = np$, we get that $\mathbb{E}[M_t|N_t] = \frac{\mu}{\mu+\omega}N_t$.

(c) What is the probability that the first woman arrives at the store before the first man?

Ans: $\mathbb{P} = \frac{\omega}{\omega+\mu}$

Hint:Here is a useful fact: if the random variables $E_1 \sim Exp(\alpha_1)$ and $E_2 \sim Exp(\alpha_2)$ are independent, then $\mathbb{P}(E_1 < E_2) = \frac{\alpha_1}{\alpha_1+\alpha_2}$ You do not need to prove this fact in your homework, but I expect that you know how to prove it.

[Food for Thought only] Could you have guessed the answer to the question in part (c) in the three limiting cases

- $\omega \to 0$ while μ is kept equal to a positive constant,$\mathbb{P} = 0$
- $\mu \to 0$ while ω is kept equal to a positive constant,$\mathbb{P} = 1$
- $\omega = \mu$? $\mathbb{P} = \frac{1}{2}$

(d) Let H denote the arrival time of the first customer (it does not matter whether it is a woman or a man). What are the probability density function and the cumulative distribution function of the random variable H? (You can answer this question without any calculations, by using your answers to the previous parts of this problem.)

Ans:

$$p(X(s+t) - X(s) = k) \quad = \tfrac{(\lambda t)^k e^{-\lambda t}}{k!}, k = 1, \lambda = \omega + \mu$$

$$p(h) = (\omega + \mu) h e^{-(\omega+\mu)h}$$
$$F(h) = \int_0^h p(h)dh = \frac{e^{-h(\omega+\mu)}(-h(\omega+\mu) + e^{h(\omega+\mu)} - 1)}{\omega + \mu}$$

(e) What is the probability that during the first three hours (i.e., in the interval [0, 3]), a total of exactly four customers have arrived at the store?

Ans:

$$p = \frac{(3(\omega+\mu))^4 e^{-3(\mu+\omega)}}{4!}$$

(f) Given that exactly four customers have arrived during the first three hours, what is the probability that all four of them were men?

Ans: $\mathbb{P}(M_t = 4 | N_t = 4) = (\frac{\mu}{\omega+\mu})^4$

[**Food for Thought only**]Could you have guessed the answer to this question in the three limiting cases from the Food for Thought part of question (c)? Does your general answer match your intuition in these limiting cases?

(g) Let T_1 denote the time of arrival of the first man at the store. Then W_{T_1} is the number of women that have arrived at the store by the time of the first mans arrival. Show that the probability distribution of the random variable W_{T_1} is

$$\mathbb{P}(W_{T_1} = k) = \frac{\omega^k \mu}{(\omega+\mu)^{k+1}} \quad \text{for } k \in \mathbb{Z}_+$$

Ans: Suppose we have k women arrived befor 1 men arrived. and these events are independent. So

$$\mathbb{P}(W_{T_1} = k) = \left(\frac{\omega}{\mu+\omega}\right)^k * \frac{\mu}{\mu+\omega} = \frac{\omega^k \mu}{(\omega+\mu)^{k+1}}$$

(h) Consider your result in part (g) in the limiting case $\mu \to 0$ (while ω is kept at a fixed strictly positive value). Does your result seem strange? How do you explain this paradox?

Ans: $\mathbb{P} = 0$ when $\mu \to 0$. Because when $\mu \to 0$, it means that the event that man occure at the store are not even happen, so the time T_1 is 0, so $\mathbb{P}(W_{T_1}) = 0$

[**Food for Thought only**] Consider the other two limiting cases as in part (c) to check if the results agree with your intuition.

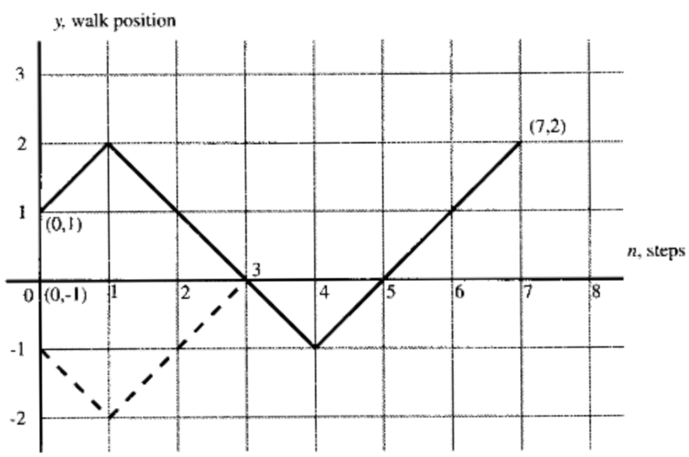
Name:Chao Jiang
MATH 5763 — Fall 2015 **HW 5** Due: 10/06/2015

Food for Thought Problem 1[1]

Consider an irreducible positive recurrent (discrete-time discrete- state space) Markov chain $\{X_n\}_{n=0}^{\infty}$, and assume that the initial state $X_0 = i$. Let $N_n(i)$ be the number of visits to state i in the first n trials, and $T_m(i)$ denote the number of trials until the mth visit to state i. Justify the relationship $\mathbb{P}(T_m(i) \geq n) = \mathbb{P}(N_n(i) \leq m)$. (Just give a convincing explanation in a couple of sentences.)

Food for Thought Problem 2

A random walk can be represented as a connected graph between coordinates (n, y), where the ordinate y is the position of the walk, and the abscissa n represents the number of steps. A walk of 7 steps which joins $(0, 1)$ and $(7, 2)$ is shown in the figure below. Suppose that a random walk starts at $(0, y_1)$ and finishes at (n, y_2), where $y_1 > 0, y_2 > 0$, and $n + y_2 y_1$ is an even number. Suppose also that the walk first visits the origin (i.e., position $y = 0$) at time $n = n_1$. Reflect that part of the path for which $n \leq n_1$ in the n-axis (see the figure), and use a reflection argument to show that the number of paths from $(0, y_1)$ to (n, y_2) which touch or cross the n-axis is equal to the number of all pats from $(0, -y_1)$ to (n, y_2). This is known as the *reflection principle*.



[1]Foot for Thought problems are for you to think about, but they do not need to be turned in with the regular homework.