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MATH 5763 — Fall 2015 **HW 4** Due: Wed, 09/23/2015

Problem 1

A pair of dice is rolled until a sum or either 5 or 7 appears. In this problem we will find the probability that a sum equal to 5 occurs first. Please follow the steps below.

(a) What is the probability that *in one individual roll of the two dice* the sum will be 5? Four your convenience, the table below represents all possible outcomes. Think of the two dice as being distinct (say, one of them is red and the other is green). Then the first number in each pair represents the outcome of the red die, and the second one represents the outcome of the green die.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Ans: $p = \frac{4}{36} = \frac{1}{9} \Longleftarrow$ 4 means that there are 4 possible combination events correspoding with a total of 36 events.

(b) What is the probability that in one individual roll of the two dice the sum will be neither 5 nor 7?

Ans: $p = 1 - \frac{4+6}{36} = \frac{13}{18} \Longleftarrow$it's complement event is the sum will be either 5 or 7.

(c) Let E_n be the event that in the sequence of rolls a 5 occurs *for the first time* on the nth roll, and no 7 has occurred before that. (In other words, E_n is the event that a 5 occurs on the nth roll and no 5 or 7 occurs in the first $n-1$ rolls.) Find the probability $P(E_n)$ of the event E_n .

Ans: $P(E_n) = (\frac{13}{18})^{n-1}(\frac{1}{9}) \Longleftarrow$ the first n-1 times will be neither 5 or 7, but the last time will be 5.

(d) Argue that the desired probability (i.e., the probability that a 5 occurs first) is equal to the infinite sum $\sum_{n=1}^{\infty} P(E_n)$

Ans: We do not know when the event will occur. All events are disjointed, so the desired probability is $\sum_{n=1}^{\infty} P(E_n)$.

(e) Find the value of the desired probability by calculating the above sum. You may need the formula for the sum of a geometric series, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, valid whenever $|q| < 1$.

Ans: $\sum_{n=1}^{\infty} P(E_n) = \frac{1}{9}\sum_{i=0}^{\infty}(\frac{13}{18})^i = \frac{1}{9} * \frac{18}{5} = \frac{2}{5}$

Problem 2

A die is rolled repeatedly. Explain in a couple of sentences why the following are Markov chains, and find their 1-step transition probability matrices **P**.

(a) The number X_n of sixes in the first n rolls.

Ans: For the initial state, it can be 6 with a probability of $\frac{1}{6}$ or it will move to next roll with probability $\frac{5}{6}$. The future state only depends on the current state, so it's markov chains. with 1-step transition probability matrics $\mathbf{P}$ as follow:

$$p_{ij} = \begin{cases} \frac{5}{6}, & \text{if } j = i, \\ \frac{1}{6}, & \text{if } j = i + 1. \\ 0, & \text{otherwise} \end{cases}$$

(b) At time n, the time X_n since the most recent six.

Ans: For the initial state, it can be 6 with a probability of $\frac{1}{6}$ then it will not move to next state, or it will move to next roll with probability $\frac{5}{6}$ which means that it did not get number 6. The future state only depends on the current state, so it's markov chains. with 1-step transition probability matrics $\mathbf{P}$ as follow:

$$p_{ij} = \begin{cases} \frac{1}{6}, & \text{if } j = i, \\ \frac{5}{6}, & \text{if } j = i + 1. \\ 0, & \text{otherwise} \end{cases}$$

Hint: The state space for both Markov chains is the set of non-negative integers, $\mathbb{Z}_+ := \{0, 1, 2, 3, \dots\}$ (but the 1-step transition probability matrices are different).

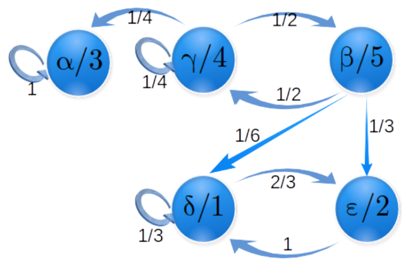
Problem 3

Consider a Markov chain whose state space consists of five states: $\alpha, \beta, \gamma, \delta, \epsilon$, and whose 1-step transition probability matrix is the following:

$$\mathbf{P} = \begin{array}{c} \begin{matrix} \alpha & \beta & \gamma & \delta & \epsilon \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{matrix}$$

(a) Draw a diagram with arrows (where each arrow from state i to state j represents a nonzero probability p_{ij}), and identify the transient and the recurrent states (do not do any computa- tions yet). You will find that two states are transient (denote the set of transient states by D), and there will be two closed and irreducible sets of recurrent states (one of them call it C_1 will consist of two states, and the other will consist of only one state call this set C_2).

Ans: $D = \{\beta, \gamma\}, C_1 = \{\delta, \epsilon\}, C_2 = \{\alpha\}$



(b) Now relabel the states $\alpha, \beta, \gamma, \delta, \epsilon$, as 1, 2, 3, 4, 5, in such a way that the $C_1 = \{1, 2\}$, $C_2 = 3$, and the states 4 and 5 to be the transient states, i.e., $D = \{4, 5\}$. In C_1, let state 1 be the state with one-step probability for transition to itself equal to $\frac{1}{3}$; in D, let state 4 be the state with nonzero one-step probability for transition to itself.

Ans: See the above figure.

(c) Carefully write down all entries in the one-step transition probability matrix $\tilde{\mathbf{P}}$ with the relabeled states. It should look like this:

$$\widetilde{\mathbf{P}} \;=\; \left(\begin{array}{c|c|c} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array}\right) ,$$

where $\mathbf{0}$ are matrices (of appropriate size) with all entries equal to zero, while the stars represent matrices that are generally not zero (but nothing more concrete can be said about them in general).

Check that $\mathbf{C}_1$ and $\mathbf{C}_2$ are stochastic matrices, while $\mathbf{T}$ is *not* a stochastic matrix.

Ans:

$$\tilde{\mathbf{P}} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: The matrix $\mathbf{C}_1$ is the same as the matrix of the 2-state Markov chain from Example 3.2.2 of Lefevbres book (on pages 81, 82), which we also discussed in Lecture 3.

Problem 4

This problem is a continuation of Problem 3.

(a) Consider the closed and irreducible set C_1 which consists of the recurrent states 1 and 2. Directly from the transition probabilities find the probabilities $\rho_{ij}^{(n)}$ of visiting state j for the first time in exactly n steps starting from state i for all possible i and j in C_1 (draw a simple diagram with these two states and think about the number of ways the first returns/visits can occur).

Solution: All the values of $\rho_{ij}^{(n)}$ for $i, j \in \{1, 2\}$ are computed in on page 82 of Lefevbres book. You do *not* need to reproduce those computations in your homework, but I expect you to understand completely Lefevbres arguments.

Ans:

$$\rho_{1,1}^{(1)} = \frac{1}{3}, \rho_{1,1}^{(2)} = \frac{2}{3} * 1 = \frac{2}{3}, \rho_{1,1}^{(n)} = 0 \text{ for } n = 3, 4, \ldots$$
$$\rho_{1,2}^{(1)} = (\frac{1}{3})^{(n-1)}(\frac{2}{3}) = \frac{2}{3^m} \text{ for } n = 1, 2, \ldots$$
$$\rho_{2,1}^{(1)} = 1, \rho_{2,1}^{(n)} = 0 \text{ for } n = 2, 3, \ldots$$
$$\rho_{2,2}^{(1)} = 0, \rho_{2,2}^{(n)} = 1 * (\frac{1}{3})^{(n-2)}(\frac{2}{3}) = \frac{2}{3^{(n-1)}} \text{ for } n = 2, 3, \ldots$$

For States 1, there are two ways to the first returns, one is from state 2 to return to State 1, and it cost 2 step $\rho_{1,1}^{(n)} = \frac{2}{3} * 1 = \frac{2}{3}$, another is revisited by itself $\rho_{1,1}^{(1)} = \frac{1}{3}$. So it does not need more than 2 steps to return to State 1.
For States 2, there are ∞ ways to the first returns considering the cycle of state 1.

(b) Use the values you obtained to compute the probabilities f_{ij} of eventually visiting state j starting from state i for all i and j in C_1. Are you surprised by the results for f_{ij}? Explain why (or why not).

Ans: All $f_{ij} = 1$, These results hold true because, here, whatever the initial state, the process is certain to eventually visit the other state considering it's a closed and irreducible class.

(c) Find $\rho_{33}^{(n)}$ and f_{33} for the only state in the set C_2. (Please explain briefly your reasoning.)Answer the same questions as in part (b).

Ans:

$$\rho_{3,3}^{(1)} = 1, \rho_{3,3}^{(n)} = 0 \text{ for } n = 2, 3, \ldots$$
$$f_{33} = \sum_{n=1}^{\infty} \rho_{3,3}^{(n)} = 1$$

Only need 1 step, and the probability is 1, So for any other $n > 1$, those probability is 0.

(d) Compute the values of ρ_{54} and ρ_{55}. (Please explain briefly your reasoning.)

Ans:

$$\rho_{5,4}^{(1)} = \frac{1}{2}, \rho_{5,4}^{(n)} = 0 \text{ for } n = 2, 3, \ldots \Longleftarrow \text{There is only one way to get State 4 from State 5}$$
$$\rho_{5,5}^{(1)} = 0, \rho_{5,5}^{(n)} = (\frac{1}{4})^{(n-1)} \text{ for } n = 2, 3, \ldots \Longleftarrow \text{ It cannot Stay at State 5 on step 1}$$

(e) Compute the values of f_{54} and f_{55}. Discuss your finding in the light of the general theory.

Ans:

$$f_{5,4} = \sum_{n=0}^{\infty} \rho_{5,4}^{(n)} = \frac{1}{2}$$
$$f_{5,5} = \sum_{n=0}^{\infty} (\frac{1}{4})^n - 1 = \frac{1}{3}$$

We can considering State 4, from State 4, one is move to State 3 with probability $\frac{1}{4}$, another one is stay at the same State 4 with probability $\frac{1}{4}$, The last one is from State 5 moved to State 4 then come back to State 5 with probabilty $\frac{1}{2} * \frac{1}{2}$.

(f) Write down the equations that the stationary distribution $\pi = (\pi_1\ \pi_2\ \pi_3\ \pi_4\ \pi_5)$ satisfies. You will very easily see from the linear system that $\pi_4 = 0$ and $\pi_5 = 0$ (but you have to obtain this from the system!). How do you explain this fact without doing any calculations?

Ans:

$$\pi = \pi\mathbf{P}, \text{where } \pi = (\pi_1\ \pi_2\ \pi_3\ \pi_4\ \pi_5)$$
$$\pi_1 = \frac{1}{3}\pi_1 + \pi_2 + \frac{1}{6}\pi_5$$
$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{3}\pi_5$$
$$\pi_3 = \pi_3 + \frac{1}{4}\pi_4$$
$$\pi_4 = \frac{1}{4}\pi_4 + \frac{1}{2}\pi_5$$
$$\pi_5 = \frac{1}{2}\pi_4$$
$$1 = \sum_{i=1}^{5} \pi_i$$

By solving the above equations, we get $\pi = (\frac{3}{5}(1-x), \frac{2}{5}(1-x), x, 0, 0)$ where $\pi_3 = x$. It's obviouse that $\pi_4 = \pi_5 = 0$, considering that State 4 and 5 are transient states. So once the moved to recurrent states, they are not be able to return.

(g) Find the most general form of a stationary distribution π. Your solution for π will be non-unique; namely, you will find that π will depend on one parameter. Discuss this in the light of the ergodic theorem.

Ans: $\pi = (\frac{3}{5}(1-x), \frac{2}{5}(1-x), x, 0, 0)$

(h) **[Food for Thought]** Can you suggest a method for computing all stationary distributions of the Markov chain in this problem without ever solving a system of five equations? Explain briefly how you are going to do it, and why your method will work.