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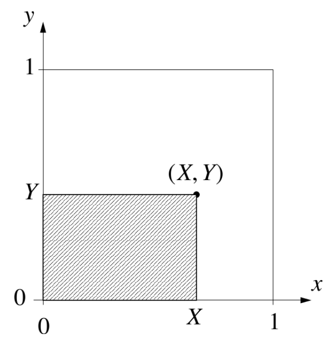
MATH 5763 — Fall 2015 **HW 3** Due: Tue, 09/15/2015

Problem 1

Let X and Y be independent random variables, each with distribution Uniform$(0,1)$. Then the point with coordinates (X,Y) is a random vector that is uniformly distributed in the unit square, i.e., its probability density function is

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let A be the random variable equal to the area of the rectangle with vortices at the points $(0,0),(0,Y),(X,0)$, and (X,Y) see the figure. Clearly, A is a continuous random variable taking values in the interval $[0,1]$.



(a) In the (x,y) plane, sketch the domain determined by the inequalities $0 \le x \le 1, 0 \le y \le 1, xy \le a$ (where a is a value between 0 and 1).

Ans: See the above figure.

(b) Show that the cumulative distribution function of A,

$$F_A(a) = \mathbb{P}(A \le a) = \mathbb{P}(XY \le a) = \iint_{xy \le a} f_{X,Y}(x,y)dx, dy,$$

is given by

$$F_A(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0], \\ a(1 - \ln a) & \text{if } a \in (0,1]. \\ 1 & \text{if } a \in [1, \infty). \end{cases}$$

Hint: What you drew in part (a) will be useful.

Ans:

$$\begin{aligned} F_A(a) \quad &= \mathbb{P}(A \le a) = \mathbb{P}(XY \le a) \\ &= \mathbb{P}(X \le \frac{a}{Y}) \\ &= \int_0^1 \mathbb{P}(X \le \frac{a}{y} | Y = y) f_Y(y) dy \Leftarrow \text{law of total probability} \\ &= \int_0^1 \mathbb{P}(X \le \frac{a}{y}) f_Y(y) dy \Leftarrow \text{X and Y are independent} \end{aligned}$$

Given that

$$\mathbb{P}(X \leq \frac{a}{y}) = \begin{cases} 1 & \text{for } 0 < y < a, \\ \frac{a}{y} & \text{for } a \leq y \leq 1. \end{cases}$$

Thus,

$$\begin{aligned} \mathbb{P}(X \leq \frac{a}{y}) \quad &= \int_0^1 \mathbb{P}(X \leq \frac{a}{y}) f_Y(y) dy \\ &= \int_0^a 1 dy + \int_a^1 \frac{a}{y} dy \\ &= a - a \ln a \end{aligned}$$

In the end, we known that

$$F_A(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0], \\ a(1 - \ln a) & \text{if } a \in (0, 1]. \\ 1 & \text{if } a \in [1, \infty). \end{cases}$$

(c) Find the probability density function, $f_A(a)$, of A. Be sure to specify $f_A(a)$ for all values of a.

Ans: We differentiate the $F_A(a)$ to get the $f_A(a)$

$$f_A(a) = \begin{cases} -\ln a & \text{if } 0 < a \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Determine the expected value $\mathbb{E}[A]$ of the area A of the random square with sides of length X and Y.

Ans:

$$\begin{aligned} \mathbb{E}[A] \quad &= \int_0^1 \int_0^1 xy f_{X,Y}(x, y) dx dy \\ &= \frac{1}{4} \end{aligned}$$

Problem 2

Let A and B be independent events in the sample space Ω, and let I_A and I_B be the corresponding indicator random variables:

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Express the following indicator random variables in terms of I_A and I_B. In each case, explain briefly your reasoning.:

(a) I_{A^c}

Ans: $I_{A^c} = 1 - I_A$

(b) $I_{A\cap B}$

Ans: $I_{A\cap B} = I_A I_B = \min\{I_A, I_B\}$

(c) $I_{A\cup B}$

Ans: $I_{A\cup B} = 1 - (1 - I_A)(1 - I_B) = \max\{I_A, I_B\}$

Hint: The easiest way to solve this problem is to come up with some guess and then check that the guess was correct.

Problem 3

Let X and Y be independent Poisson random variables with respective parameters λ and μ, *i.e.*,

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad p_Y(k) = e^{-\mu}\tfrac{\mu^k}{k!}, \quad k = 0, 1, 2, \ldots .$$

Show that:

(a) $X + Y$ is Poisson with parameter $\lambda + \mu$;

Ans:

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X+Y=k, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i, X=i) \\
&= \sum_{i=0}^{k} e^{-\mu}\frac{\mu^{k-i}}{(k-i)!}e^{-\lambda}\frac{\lambda^i}{i!} \\
&= e^{-(\mu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\mu^{k-i}\lambda^i \\
&= e^{-\mu+\lambda}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\mu^{k-i}\lambda^i \\
&= \frac{(\mu+\lambda)^k}{k!}e^{-(\mu+\lambda)}
\end{aligned}$$

So $X + Y$ is Poisson with parameter $\lambda + \mu$

(b) the conditional distribution of X given that $X + Y = n$ is binomial, and find its parameters.

Ans:

$$
\begin{aligned}
P(X=k|X+Y=n) &= \frac{P(X=k)P(Y=n-k)}{P_{X+Y}(n)} \\
&= \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}}{\frac{(\mu+\lambda)^n}{n!}e^{-(\mu+\lambda)}} \\
&= \frac{n!}{k!(n-k)!}\frac{\lambda^k\mu^{n-k}}{(\mu+\lambda)^n} \\
&= \binom{n}{k}(\frac{\lambda}{(\mu+\lambda)})^k(\frac{\mu}{\lambda+\mu})^{n-k}
\end{aligned}
$$

So the distribution is binomial with parameters $(\frac{\lambda}{\lambda+\mu}, n)$

Problem 4

Let X be a stationary discrete-time discrete-state space Markov chain with state space S consisting of two states, 0 and 1, and let the 1-step transition probability matrix of the stochastic process be

$$
\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}, \tag{1}
$$

where $p_{ij} = \mathbb{P}(X_{n+1} = j|X_n = i)$. Assume that you do not know the exact value of the initial value X_0 of the MC X, but you know that

$$
\mathbb{P}(X_0 = 0) = \frac{1}{5} \tag{2}
$$

(a) Find the p.m.f. p_{X_0} of the initial state X_0 of the MC.

Ans:

$$
p_{X_0} = \begin{cases} \frac{1}{5}, & X_0 = 0, \\ \frac{4}{5}, & X_0 = 1. \end{cases}
$$

(b) Find $\mathbb{E}[X_0]$ and Var X_0.

Ans:

$$
\begin{aligned}
\mathbb{E}[X_0] &= 0 * \frac{1}{5} + 1 * \frac{4}{5} = \frac{4}{5} \\
\mathrm{Cov}[X_0] &= (\frac{4}{5})^2 * \frac{1}{5} + (\frac{1}{5})^2 * \frac{4}{5} = \frac{4}{25}
\end{aligned}
$$

(c) Find the p.m.f. p_{X_1} of the state X_1 of the MC at time 1.

Ans:

$$
p_{X_1} = \begin{cases} \frac{1}{5} * \frac{1}{3} + \frac{4}{5} * 1 = \frac{13}{15}, & X_0 = 0, \\ \frac{1}{5} * \frac{2}{3} = \frac{2}{15}, & X_0 = 1. \end{cases}
$$

(d) Find $\mathbb{E}[X_1]$.

Ans:

$$\mathbb{E}[X_1] \quad = 0 * \frac{13}{15} + 1 * \frac{2}{15} = \frac{2}{15}$$

(e) Find the p.m.f. p_{X_2} of the state X_2 of the MC at time 2.

Ans:

$$\begin{aligned} p_{X_2} &= p_{X_0}\mathbf{P}^2 \\ &= \begin{cases} \frac{19}{45}, & X_0 = 0, \\ \frac{26}{45}. & X_0 = 1. \end{cases} \end{aligned}$$

Problem 5

Consider the MC from Problem 4. Let

$$\rho_{ij}^{(n)} := \mathbb{P}(X_n = j, X_{n-1} \neq j, \ldots, | X_0 = i), \quad n \geq 1, \quad i, j \in \{0, 1\}$$

be the probability of moving to state j, from the initial state i, for the first time at the nth transition. Recall the relations

$$p_{ij}^{(n)} = \sum_k \rho_{ij}^{(k)} p_{jj}^{(n-k)} \tag{3}$$

where $p_{ij}^{(n)}$ are the entries of the n-step transition probability matrix $\mathbf{P}^{(n)}$. Assume (quite reasonably) that $\mathbf{P}^{(0)} = \mathbf{I}$ (the identity matrix).

(a) Compute explicitly $\mathbf{P}^{(0)}$, $\mathbf{P}^{(1)}$,$\mathbf{P}^{(2)}$,and $\mathbf{P}^{(3)}$

Ans:

$$\mathbf{P}^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{P}^{(1)} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix} \mathbf{P}^{(2)} = \begin{bmatrix} \frac{7}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \mathbf{P}^{(3)} = \begin{bmatrix} \frac{13}{27} & \frac{14}{27} \\ \frac{7}{9} & \frac{2}{9} \end{bmatrix}$$

(b) Use (3) to compute the value of $\rho_{00}^{(1)}$.

Ans:

$$p_{00}^{(1)} = \rho_{00}^{(1)} p_{00}^{(0)} \Rightarrow \rho_{00}^{(1)} = \frac{1}{3}$$

(c) Use (3) and the value of $\rho_{00}^{(1)}$(found in (b)) to compute the value of $\rho_{00}^{(2)}$.

Ans:

$$p_{00}^{(2)} = \rho_{00}^{(1)} p_{00}^{(1)} + \rho_{00}^{(2)} p_{00}^{(0)} \Rightarrow \rho_{00}^{(2)} = \frac{2}{3}$$

(d) Use (3) and the values of $\rho_{00}^{(1)}$ and $\rho_{00}^{(2)}$ (found in (b) and (c)) to compute the value of $\rho_{00}^{(3)}$.

Ans:

$$p_{00}^{(3)} = \rho_{00}^{(1)} p_{00}^{(2)} + \rho_{00}^{(2)} p_{00}^{(1)} + \rho_{00}^{(3)} p_{00}^{(0)} \Rightarrow \rho_{00}^{(3)} = 0$$

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Food for Thought Problem 1[1]

Prove the recursive relation (3).

Solution: Let $i \in \{0,1\}, j \in \{0,1\}$, and $n \geq 1$ be fixed. Notice that the events

$$A_k := \{\text{the MC reaches state } j \text{ for the first time in exactly } k \text{ steps}\}.$$

Clearly, for a given n, the time k when the MC reaches state j for the first time (starting from state i) can take values $1, 2, ..., n$. It is obvious that the events A_k form a partition of the sample space Ω (why?):

$$\bigcup_{k=1}^{n} A_k = \Omega, \quad A_k \cap A_{k'} = \emptyset \text{ for } k \neq k' \tag{4}$$

Note that

$$\rho_{ij}^{(n)} = \mathbb{P}(A_k | X_0 = i) \tag{5}$$

Using (4) and (5), we obtain

$$\begin{aligned}
p_{ij}^{(n)} &= \mathbb{P}(X_n = j | X_0 = i) = \mathbb{P}(\{X_n = j\} \cap \Omega | X_0 = i) \\
&= \mathbb{P}\left(\{X_n = j\} \cap \bigcup_{k=1}^{n} A_k \middle| X_0 = i\right) \\
&= \mathbb{P}\left(\bigcup_{k=1}^{n} (\{X_n = j\} \cap A_k) \middle| X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}(\{X_n = j\} \cap A_k | X_0 = i) \\
&= \sum_{k=1}^{n} \frac{\mathbb{P}(\{X_n = j\} \cap A_k \cap \{X_0 = i\})}{\mathbb{P}(X_0 = i)} \\
&= \sum_{k=1}^{n} \frac{\mathbb{P}(\{X_n = j\} \cap A_k \cap \{X_0 = i\})}{\mathbb{P}(A_k \cap \{X_0 = i\})} \frac{\mathbb{P}(A_k \cap \{X_0 = i\})}{\mathbb{P}(X_0 = i)} \\
&= \sum_{k=1}^{n} \mathbb{P}(X_n = j | A_k \cap \{X_0 = i\}) \, \mathbb{P}(A_k | X_0 = i) \\
&= \sum_{k=1}^{n} \mathbb{P}(X_n = j | A_k) \, \mathbb{P}(A_k | X_0 = i) \\
&= \sum_{k=1}^{n} \mathbb{P}(X_n = j | X_k = j) \, \mathbb{P}(A_k | X_0 = i) \\
&= \sum_{k=1}^{n} \mathbb{P}(X_{n-k} = j | X_0 = j) \, \mathbb{P}(A_k | X_0 = i) \\
&= \sum_{k=1}^{n} p_{ij}^{(n-k)} \rho_{ij}^{(n)}
\end{aligned}$$

[1]Foot for Thought problems are for you to think about, but they do not need to be turned in with the regular homework.