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MATH 5763 — Fall 2015 **HW 2** Due: Wed, 09/09/2015

Problem 1

Let A and B be events with $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.4$. Find the conditional probability $\mathbb{P}(A|B)$ in the following cases:

(a) A and B are mutually exclusive (i.e., disjoint);

Ans: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ Given that $\mathbb{P}(A \cap B) = 0$, then $\mathbb{P}(A|B) = 0$.

(b) $\mathbb{P}(A \cap B) = 0.1$;

Ans: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.1}{0.4} = \frac{1}{4}$.

(c) A implies B (i.e., every time A occurs, B also occurs).

Ans: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$. Given that $\mathbb{P}(B|A) = 1$, then $\mathbb{P}(A|B) = \frac{3}{4}$

Problem 2

Let X be a continuous RV uniformly distributed over the interval $[0, 1]$, i.e., the p.d.f. of X is

$$f_X(x) = \begin{cases} 1, & \text{if x} \in [0,1], \\ 0, & otherwise. \end{cases}$$

Let X_1 and X_2 be independent continuous RVs modeled after the RV X, i.e., $f_{X_1}(x) = f_{X_2}(x) = f_X(x)$ for any $x \in \mathbb{R}$.

(a) Derive the expression for the c.d.f. $F_X(x)$ of the random variable X.

Ans:

$$F_X(x) = \begin{cases} 0, & \text{if x} < 0, \\ x, & \text{if x} \in [0,1], \\ 1, & \text{if x} > 1. \end{cases}$$

(b) Prove that the minimum, $X_{min} = \min\{X_1, X_2\}$, of the RVs X_1 and X_2 has c.d.f.

$$F_{X_{min}}(x) = 1 - [1 - F_X(x)]^2$$

Please point out at which point in your proof you used the independence of X_1 and X_2.

Ans:

$$\begin{aligned} F_{X_{min}}(x) \quad &= \mathbb{P}(X_{min} \le x) \Longleftarrow \exists X \le x \\ &= 1 - \mathbb{P}(X_{min} > x) \text{ Equivalent to one minus the probability that } \forall X > x \\ &= 1 - \mathbb{P}(X_1 > x, X_2 > x) \\ &= 1 - \mathbb{P}(X_1 > x)\,\mathbb{P}(X_2 > x) \Longleftarrow \text{Considering independent of } X_1, X_2 \\ &= 1 - (1 - \mathbb{P}(X_1 \le x))(1 - \mathbb{P}(X_2 \le x)) \\ &= 1 - (1 - F_X(x))(1 - F_X(x)) \\ &= 1 - [1 - F_X(x)]^2 \end{aligned}$$

(c) Use your result for the c.d.f. of X_{min} from part (b) to find the p.d.f. $f_{X_{min}}(x)$ of X_{min}.

Ans: Given the result of part (b), we can get the c.d.f of X_{min} below:

$$F_{X_{min}}(x) = \begin{cases} 0, & \text{if x} < 0, \\ 1 - [1-x]^2, & \text{if x} \in [0,1], \\ 1, & \text{if x} > 1. \end{cases}$$

So the p.d.f of X_{min} is the derivative of c.d.f.

$$f_{X_{min}}(x) = \frac{d(F_{X_{min}}(x))}{dx} = \begin{cases} 0, & \text{if x} < 0, \\ 2(1-x), & \text{if x} \in [0,1], \\ 0, & \text{if x} > 1. \end{cases}$$

(d) What can you say about the expectation of X_{min} *without doing any calculations*? (Not the exact value, just say *something reasonable*, and explain how you came to this conclusion.)

Ans: The $X_{min} < \frac{1}{2}$ which is the expectation of X. considering the $X_{min} = \{X_1, X_2\}$, it is reasonable that the value of $\mathbb{E}[X_{min}] < \frac{1}{2}$.

(e) Now compute the exact value of $\mathbb{E}[X_{min}]$.

Ans:

$$\mathbb{E}(X_{min}) = \int_{-\infty}^{\infty} x f_{x_{min}}(x) dx = x^2 - \frac{2}{3}x^3 \Big|_{x=0}^{x=1} = \frac{1}{3}$$

Problem 3

Let X be a continuous RV uniformly distributed over the interval $[0,1]$ as in Problem 1. Define the continuous RV Y to be a function of the RV X defined as

$$Y = -\ln X.$$

(a) Find the interval of values where Y takes values. (Do not worry that X can take value 0 this event occurs with probability 0, so you can ignore it. Think of X as taking values in the interval $(0,1]$.)

Ans: The value of $\ln X$ between $(0,1]$ is $(-\infty, 0]$. So the $0 \leq -\ln X < \infty$. Thus the range of Y is $[0, \infty)$

(b) Directly from the definition of the c.d.f., $F_Y(y) = P(Y \leq y)$, of the RV Y, show that

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1 - e^{-y}, & \text{if } y > 0. \end{cases}$$

(note that I did not write anything about $F_Y(0)$ since Y is a continuous RV, $\mathbb{P}(Y = 0) = 0$, so that the value of $F_Y(0)$ is not important). You may that the following events are the same:

$$\{Y \leq y\} = \{-\ln X \leq y\} = \{\ln X \geq -y\} = \{X \geq e^{-y}\} = \{X < e^{-y}\}^c.$$

Ans: Given the c.d.f. of X from probelm 2, we can get the below equations:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= 1 - \mathbb{P}(X < e^{-y}) \Longleftarrow \text{From the hint we can get} \\ &= \begin{cases} 0, & \text{if } y < 0, \Longleftarrow y < 0, \text{ then } e^{-y} > 1, \\ 1 - e^{-y}, & \text{if } y > 0. \Longleftarrow y > 0, \text{ then } 0 < e^{-y} < 1. \end{cases} \end{aligned}$$

(c) Use your result from part (b) to find $f_Y(y)$.

Ans: Taken the derivative of $F_Y(y)$ with respect to y, we get:

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} \\ &= \begin{cases} 0, & \text{if } y < 0, \\ e^{-y}, & \text{if } y > 0. \end{cases} \end{aligned}$$

(d) Now use the formula for the p.d.f. of a function, $Y = g(X)$, of a RV, namely

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|,$$

to find $f_Y(y)$ (of course, you should obtain the same result as in part (c)).

Ans: Given the p.d.f of X from Problem 2, we can get that:

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \\ &= f_X(e^{-y})\left|\frac{de^{-y}}{dy}\right| \\ &= \begin{cases} 0, & \text{if } y < 0, \\ e^{-y}, & \text{if } y > 0 \Rightarrow f_X(e^{-y}) = 1. \end{cases} \end{aligned}$$

Problem 4

The joint p.m.f. $p_{X,Y}(x_k, y_m) = \mathbb{P}(X = x_k, Y = y_m)$ of the discrete RVs X and Y has values given in the table below.

	Y=1	Y=3	Y=4
X=5	0	$\frac{1}{15}$	$\frac{2}{15}$
X=7	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(a) Find the marginal p.m.f.s $p_X(x_k)$ and $p_Y(y_m)$.

Ans:

$$p_X(x_k = 5) = \frac{3}{15}, p_X(x_k = 7) = \frac{12}{15};$$
$$p_Y(y_m = 1) = \frac{3}{15}, p_Y(y_m = 3) = \frac{5}{15}, p_Y(y_m = 4) = \frac{7}{15}.$$

(b) Find the expected value $\mathbb{E}[X]$ of X.

Ans:

$$\mathbb{E}[x] = 5 * \frac{3}{15} + 7 * \frac{12}{15} = \frac{33}{5}$$

(c) Compute the conditional p.m.f.s $p_{X|Y}(x_k|y_m) = \mathbb{P}(X = x_k|Y = y_m)$ for $y_m = 1, 3, 4$.

Ans:

$$\begin{aligned}
p_{X|Y}(x_k = 5|y_m = 1) &= \frac{p_{X,Y}(x_k = 5, y_m = 1)}{p_Y(y_m = 1)} = 0,\\
p_{X|Y}(x_k = 7|y_m = 1) &= \frac{p_{X,Y}(x_k = 7, y_m = 1)}{p_Y(y_m = 1)} = 1;\\
p_{X|Y}(x_k = 5|y_m = 3) &= \frac{p_{X,Y}(x_k = 5, y_m = 3)}{p_Y(y_m = 3)} = \frac{1}{5},\\
p_{X|Y}(x_k = 7|y_m = 3) &= \frac{p_{X,Y}(x_k = 7, y_m = 3)}{p_Y(y_m = 3)} = \frac{4}{5};\\
p_{X|Y}(x_k = 5|y_m = 4) &= \frac{p_{X,Y}(x_k = 5, y_m = 4)}{p_Y(y_m = 4)} = \frac{2}{7},\\
p_{X|Y}(x_k = 7|y_m = 4) &= \frac{p_{X,Y}(x_k = 7, y_m = 4)}{p_Y(y_m = 4)} = \frac{5}{7}.
\end{aligned}$$

(d) Find the conditional expectations $\mathbb{E}[X|Y = y_m]$ for $y_m = 1, 3, 4$.

Ans:

$$\begin{aligned}
\mathbb{E}[X|y_m = 1] &= 0 * 5 + 1 * 7 = 7;\\
\mathbb{E}[X|y_m = 3] &= \frac{1}{5} * 5 + \frac{4}{5} * 7 = \frac{33}{5};\\
\mathbb{E}[X|y_m = 4] &= \frac{2}{7} * 5 + \frac{5}{7} * 7 = \frac{45}{7}.
\end{aligned}$$

(e) The quantity $\mathbb{E}[X|Y]$ can be considered as a random variable which is a function of Y, and written as a linear combination of the indicator functions of the sets $\{Y = y_m\} = Y^{-1}(y_m)$:

$$\mathbb{E}[X|Y] = \sum_m \mathbb{E}[X|Y = y_m] I_{Y^{-1}(y_m)}.$$

Use your result from part (d) to write $\mathbb{E}[X|Y]$ in this form (using concrete numbers).

$$\begin{aligned}
\mathbb{E}[X|Y] \quad &= \mathbb{E}[X|Y = 1]I_{Y^{-1}(1)} + \mathbb{E}[X|Y = 3]I_{Y^{-1}(3)} + \mathbb{E}[X|Y = 4]I_{Y^{-1}(4)}\\
&= 7 * I_{Y^{-1}(1)} + \frac{33}{5} * I_{Y^{-1}(3)} + \frac{45}{7} * I_{Y^{-1}(4)}
\end{aligned}$$

(f) Use the representation of $\mathbb{E}[X|Y]$ from part (e) to find its expectation, $\mathbb{E}[\mathbb{E}[X|Y]]$. According to the so-called tower rule, the following equality should hold

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

(but here you have to compute $\mathbb{E}[\mathbb{E}[X|Y]]$ directly).

Ans:

$$\mathbb{E}_Y[\mathbb{E}[X|Y]] = \frac{3}{15} * 7 + \frac{5}{15} * \frac{33}{5} + \frac{7}{15} * \frac{45}{7} = \frac{33}{5}$$

Problem 5

A frog lays Y eggs, where Y is a Poisson RV with parameter $\lambda > 0$:

$$Y \sim Poi(\lambda)$$

i.e., the p.m.f. of Y is

$$p_Y(n) = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n = 0, 1, 2, ...,$$

where $0! := 1$.

Each egg survives independently of the survival of the other eggs, with probability $p \in (0, 1)$. From this one can derive that the number X of surviving eggs is a binomial RV with parameters Y and p:

$$X \sim Bin(Y, p).$$

This means that the RV X, conditioned on the event $\{Y = n\}$, is binomial with parameters n and p:

$$X|\{Y = n\} \sim Bin(n, p),$$

i.e., that the conditional p.m.f. of X conditioned on the value of Y is

$$p_{X|Y}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, ..., n,$$

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ are the binomial coefficients.

Show by a direct calculation of the p.m.f. of X that

$$X \sim Poi(\lambda p).$$

Hint: Use equation (7) derived in Food For Thought Problem 2. You may also find some of the following facts useful:

- series expansion of e^t :

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}, \quad t \in \mathbb{R};$$

- binomial formula:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k};$$

- geometric series:

$$\frac{1}{1-q} = \sum_{k=0}^{\infty} q^k, \quad |q| < 1.$$

Ans: Computing the marginal distribution of $\mathbb{P}(X)$

$$\begin{aligned}
\mathbb{P}(X) \quad &= \sum_{n=k}^{\infty} \mathbb{P}_{X,Y}(x,y) \\
&= \sum_{n=k}^{\infty} \mathbb{P}_{X|Y}(x|y)\, \mathbb{P}_Y(y) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \left(\frac{e^{-\lambda}\lambda^n}{n!} \right) \\
&= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{((1-p)\lambda)^{n-k}}{(n-k)!} \\
&= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{m=0}^{\infty} \frac{((1-p)\lambda)^{m}}{(m)!} \Leftarrow \; m = (n-k) \\
&= \frac{(\lambda p)^k e^{-\lambda}}{k!} * e^{(1-p)\lambda} \Leftarrow \; \text{according to the above hint of series expansion of } e^t \\
&= \frac{(\lambda p)^k e^{-\lambda p}}{k!}
\end{aligned}$$

So we get that X has a $Poi(\lambda p)$ distribution

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Food for Thought Problem 1[1]

Let Y be a discrete RV taking values y_m, where the index m runs over a discrete set (finite or infinite). For a given m, consider the pre-image of y_m , i.e., the set

$$Y^{-1}(y_m) = \{\omega \in \Omega Y(\omega) = y_m\}, \text{ which is often written as } \{Y = y_m\}. \tag{1}$$

(a) Convince yourself that when m runs over all possible values, the sets $\{Y = y_m\}$ form a partition of the sample space Ω, i.e.,

$$\bigcup_m Y^{-1}(y_m) = \Omega, \qquad Y^{-1}(y_m) \cap Y^{-1}(y_n) = \emptyset \text{ for } m \neq n. \tag{2}$$

Ans: One possible value can be view as a element event of the whole space Ω.

(b) Let $A \subset \Omega$ be an event. Recall that the *indicator function*,$I_A : \Omega \to \mathbb{R}$, of an event A is defined as

$$I_A(\omega) = \begin{cases} 0, & \text{if } \omega \notin A, \\ 1, & \text{if } \omega \in A. \end{cases}$$

If $I_A^{-1} : \{0, 1\} \to \Omega$ stands for the inverse of I_A^{-1} (of course, I_A^{-1} can take only the numbers 0 or 1 as an argument), then obviously

$$\begin{aligned} I_A^{-1}(0) = \{\omega \in \Omega : I_A(\omega) = 0\} &= A^c \\ I_A^{-1}(1) = \{\omega \in \Omega : I_A(\omega) = 1\} &= A \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(E_A^{-1}(0)) &= \mathbb{P}(\{\omega \in \Omega : I_A(\omega) = 0\}) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A), \\ \mathbb{P}(E_A^{-1}(1)) &= \mathbb{P}(\{\omega \in \Omega : I_A(\omega) = 1\}) = \mathbb{P}(A) \end{aligned} \tag{3}$$

For a given m, consider the indicator function $I_Y^{-1}(y_m) : \Omega \to \{0, 1\}$ of the set $Y^{-1}(y_m)$. This is a discrete RV variable satisfying

$$I_Y^{-1} y_m(\omega) = \begin{cases} 0, & \text{if } \omega \notin Y^{-1}(y_m), \text{ i.e., } \omega \notin Y^{-1}(y_m), \\ 1, & \text{if } \omega \in Y^{-1}(y_m), \text{ i.e., } \omega \in Y^{-1}(y_m), \end{cases}$$

Convince yourself that the RV Y can be written as a linear combination of the indicator functions of the sets $Y^{-1}(y_m)$ as m runs over all allowed values, as follows:

$$Y = \sum_m y_m I_{Y^{-1}(y_m)} \tag{4}$$

(simply note that $I_{Y^{-1}}(y_m)(\omega)$ is equal to 1 exactly if $Y(\omega) = y_m$).

Ans: From the above statments, we know that $I_{Y^{-1}(y_m)}$ stands for the probabilty of $Y = y_m$. So we have different probabilty to get all different values y_m.

[1]Foot for Thought problems are for you to think about, but they do not need to be turned in with the regular homework.

(c) Convince yourself that the p.m.f.

$$p_{I_A}(y) = \mathbb{P}(I_A = y) = \mathbb{P}(I_A^{-1}(y)) \quad \text{for } y \in \{0, 1\}$$

of the indicator function I_A is equal to (recall (3))

$$p_{I_A}(y) = \mathbb{P}(I_A^{-1}(y)) = \begin{cases} \mathbb{P}(A^c) = 1 - \mathbb{P}(A) & \text{if } y = 0, \\ \mathbb{P}(A) & \text{if } y = 1, \end{cases} \tag{5}$$

Ans: We know that $\mathbb{P}(I_A(y = 1))$ stands for the probability of event A occures. also $\mathbb{P}(I_A(0)$ means the probabilty of event A does not happen.

(d) Directly from the definition of expectation and from (5), show that

$$\mathbb{E}[I_A] = \mathbb{P}(A) \tag{6}$$

Ans:

$$\begin{aligned} \mathbb{P}(I_A = 0) &= 1 - \mathbb{P}(A) \\ \mathbb{P}(I_A = 1) &= \mathbb{P}(A) \\ \mathbb{E}[I_A] &= 1 * \mathbb{P}(A) + 0 * (1 - \mathbb{P}(A)) \\ &= \mathbb{P}(A) \end{aligned}$$

(e) Take expectation of both sides of (4) and use the linearity property of expectation (i.e., the fact that $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$) and (6) to obtain

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_m y_m I_{Y^{-1}(y_m)}\right] = \sum_m y_m\,\mathbb{E}[I_{Y^{-1}(y_m)}] = \sum_m y_m\,\mathbb{P}(Y^{-1}(y_m)) = \sum_m y_m p_Y(y_m),$$

which coincides with the definition of $\mathbb{E}[Y]$, as it should.

Food for Thought Problem 2

Let X and Y be discrete RVs taking values x_k and y_m, where the indices k and m run over discrete sets (finite or infinite).
Use the notation (1) and the fact (2) that the sets $\{Y = y_m\}$ form a partition of the sample space Ω, to convince yourselves that the following derivation of the p.m.f. of the RV X through conditioning on the RV Y is correct:

$$\begin{aligned} p_X(x_k) &= \mathbb{P}(X = x_k) = \mathbb{P}(\{X = x_k\} \cap \Omega) \\ &= \mathbb{P}\left(\{X = x_k\} \cap \bigcup_m \{Y = y_m\}\right) \\ &= \mathbb{P}\left(\bigcup_m \{X = x_k, Y = y_m\}\right) \\ &= \sum_m \mathbb{P}(X = x_k, Y = y_m) \\ &= \sum_m \mathbb{P}(X = x_k | Y = y_m)\,\mathbb{P}(Y = y_m) \\ &= \sum_m p_{X|Y}(x_k|y_m) p_Y(y_m) \end{aligned} \tag{7}$$