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MATH 5763 — Fall 2015 **HW 1** Due: Wed, 09/02/2015

Problem 1

An apparatus consists of three modules, labeled 1, 2, and 3. For the apparatus to be working, module 1 must be working, and at least one of the other two modules, 2 and 3, must be working. Let E_i stands for the event module #i is working, and let $P(E_i) = p_i$. The events E_i are independent (i.e., module #i is working independently of whether the other modules are working).

(a) Express the event W := the apparatus is working in terms of the events E_1, E_2, and E_3.

$$W = E_1 \cap (E_2 \cup E_3)$$

(b) Find the probability that the apparatus is working.

$$P(W) = p_1(p_2 + p_3 - p_2p_3)$$

(c) Find the probability that the apparatus is working if module #3 has burned (i.e., is not working)?

$$P(W|\overline{E_3}) = p_1p_2$$

(d) What is the probability that module #3 has burned (i.e., is not working), if the apparatus is working?

$$P(\overline{E_3}|W) = \frac{P(\overline{E_3})P(W|\overline{E_3})}{P(W)} = \frac{(1-p_3)p_2}{p_2+p_3-p_2p_3}$$

Problem 2

Amal and Chao go target shooting together. Both shoot at the same target at the same time. Suppose that Amal hits the target with probability $p_A = 0.7$, whereas Chao, independently, hits the target with probability $p_C = 0.4$.

(a) Determine the probability that the target is hit (by at least one bullet).

$$p(T) = p_A + p_C - p_Ap_C = 0.82$$

(b) Given that the target is hit, what is the probability that Chao hit it?

$$p(C|T) = \frac{p_C}{p(T)} = \frac{20}{41}$$

(c) Find the probability that exactly one shot hits the target.

$$p(O) = p_A * (1 - p_C) + p_C(1 - p_A) = 0.54$$

(d) Given that exactly one shot hits the target, what is the probability that it was Chaos shot?

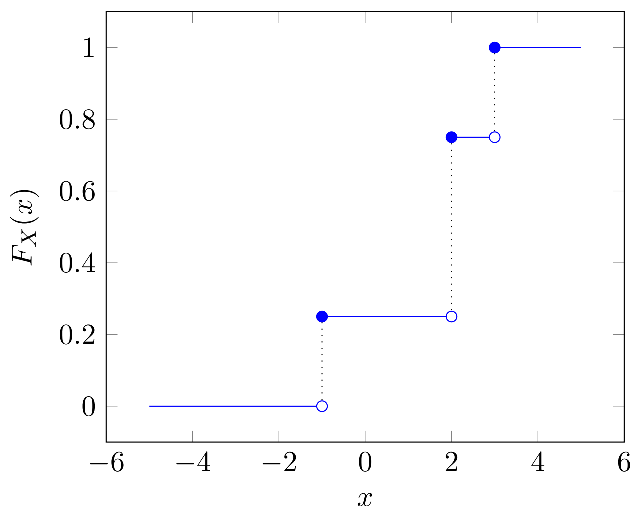
$$p(C|O) = \frac{p_C * (1 - p_A)}{p(O)} = \frac{2}{9}$$

Problem 3

The cumulative distribution function F_X of the discrete random variable X is the following:

$$F_X(x) = \begin{cases} 0, & x < -1, \\ \frac{1}{4}, & -1 \leq x < 2, \\ \frac{3}{4}, & 2 \leq x < 3, \\ 1, & 3 \leq x. \end{cases}$$

(a) Plot the graph of F_X.



(b) What are the values that the random variable X takes?

$$\{-1, 2, 3\}$$

(c) Find the probability mass function p_X of the random variable X.

$$p_X(x) = \begin{cases} \frac{1}{4}, & x = -1, \\ \frac{1}{2}, & x = 2, \\ \frac{1}{4}, & x = 3. \end{cases}$$

(d) The *expectation*, $\mathbb{E}[X]$, of the random variable X is defined as

$$\mathbb{E}[X] = \sum_k x_k p_X(x_k)$$

where the summation is over all values x_k that X can take. Find the value of $\mathbb{E}[X]$.

$$\mathbb{E}[X] = -1 * \frac{1}{4} + 2 * \frac{1}{2} + 3 * \frac{1}{4} = \frac{3}{2}$$

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Problem 4

Let $X_1, ..., X_n$ be n independent random variables, and let $F_{X_n}(x)$ be the cumulative distribution function of the random variable X_n, defined as the probability of the event $\{X_n \leq x\}$:

$$F_{X_n}(x) = \mathbb{P}(X_n \leq x).$$

Let Y be a random variable that is equal to the maximum of the random variables $X_1, ..., X_n$:

$$Y = \max\{X_1, ..., X_n\}.$$

(a) Let x be a real number. Explain why the event $\{Y \leq x\}$ can be written as the intersection of the events $\{X1 \leq x\}, ..., \{X_n \leq x\}$:

Ans:
$\{Y \leq x\}$ means that evnets $\{X_1 \leq x\}$, ... $\{X_n \leq x\}$ satisfy at the same time. Also we know that these n variables are independent. So

$$\{Y \leq x\} = \bigcap_{j=1}^{n} \{X_j \leq x\}.$$

(b)) Use the result of part (a) to show that the c.d.f. of Y is the product of the c.d.f.s of the random variables $X_1, ..., X_n$:

$$F_Y(x) = \prod_{j=1}^{n} F_{X_n}(x).$$

Please indicate clearly where you have used the independence of $X_1, ..., X_n$.

Ans:

$$\begin{aligned} F_Y(x) \quad &= \mathbb{P}(Y \leq x) \\ &= \mathbb{P}(\max\{x_1, ..., x_n\} \leq x) \Longleftarrow \text{ due to the result of part (a)} \\ &= \mathbb{P}(\bigcap_{j=1}^{n} \{X_j \leq x\}) \Longleftarrow \text{ due to the independent of random variables } X_j \\ &= \prod_{j=1}^{n} F_{X_n}(x). \end{aligned}$$

we now get that $F_Y(x) = \prod_{j=1}^{n} F_{X_n}(x)$.

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Food for Thought Problem 1[1]

Let A and B be events with $\mathbb{P}(A) = \frac{3}{4}$, and $\mathbb{P}(B) = \frac{1}{3}$. Convince me that

$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$$

by a brief verbal explanation and/or a clear picture.

Ans:
If events A, B are independent, then $\mathbb{P}(A \cup B) > 1$, so these two events must have intersection. while, the minimum part of the intersection is $\frac{3}{4} + \frac{1}{3} - x = 1 \Rightarrow x = \frac{1}{12}$. another situation is when event A occures, event B must occures, in other words, $B \subsetneq A$. So the events A and B happens together with the probability:$\mathbb{P}(A \cap B) = p(B)p(B|A) = \frac{1}{3}$.

Food for Thought Problem 2

Suppose that two fair dice a red one and a green one are rolled. Consider the events

$A =$ "The red die shows an odd number."

$B =$ "The green die shows an odd number."

$C =$ "The sum of the numbers on the two dice is odd."

Show that the events A, B, and C are pairwise independent, but not independent.

Ans:
First, we know that:

$$\begin{aligned} P[A] = P[B] = P[C] &= \frac{1}{2} \\ P[A \cap B] = P[A \cap C] = P[B \cap C] &= \frac{1}{4} \end{aligned}$$

So that A, B, C are pairwise independent.While

$$P[A \cap B \cap C] = P[A \cap \bar{B}] = P[\bar{A} \cap B] = \frac{1}{4}$$

Here $\bar{B}$ means green die shows an even number, and $\bar{A}$ means red die shows an even number with $p(\bar{A}) = p(\bar{B}) = \frac{1}{2}$. So they are not independent as a triplet.

[1]Foot for Thought problems are for you to think about, but they do not need to be turned in with the regular homework.