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MATH 5763 — Fall 2015 **Midterm Exam** Due: 11/02/2015

Problem 1

Ans:

(a) This chain is irreducible. Because all of the states are communicate.

(b)

$$\mathbf{P_h} = \begin{pmatrix} 1-\rho h & \rho h & 0 & 0 & 0 & \cdots \\ i\mu h & 1-(\rho+i\mu)h & \rho h & 0 & 0 & \cdots \\ 0 & i\mu h & 1-(\rho+i\mu)h & \rho h & 0 & \\ 0 & \cdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & i\mu h & 1-(\rho+i\mu)h & \rho h \end{pmatrix}$$

$$\mathbf{G} = (v_{ij}) = \frac{d\mathbf{P_h}}{dh} \begin{pmatrix} -\rho & \rho & 0 & 0 & 0 & \cdots \\ i\mu & -(\rho+i\mu) & \rho & 0 & 0 & \cdots \\ 0 & i\mu & -(\rho+i\mu) & \rho & 0 & \\ 0 & \cdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & i\mu & -(\rho+i\mu) & \rho \end{pmatrix}$$

From the above matrix we known that $\sum_{j=0}^{\infty} v_{ij} = 0$ for each $i \in \mathbb{Z}_+$

(c)

$$\begin{aligned} p_0'(t) &= \lim_{h\to 0} \frac{p_0(t+h) - p_0(t)}{h} \\ p_0(t+h) &= p_1(t)\mu h + (1-\rho h)p_0(t) \\ \Rightarrow p_0'(t) &= p_1(t)\mu - \rho p_0(t) \\ p_j'(t) &= \lim_{h\to 0} \frac{[1-(\rho+j\mu)]p_j(t) + \rho h p_{j-1}(t) + j\mu h p_{j+1}(t)}{h} \\ &= -(\rho+j\mu)p_j(t) + \rho p_{j-1}(t) + j\mu p_{j+1}(t) \end{aligned}$$

(d)

$$\begin{aligned} \Delta(\xi, 0) &= \sum_{j=0}^{\infty} p_j(0)\xi^j \\ &= \sum_{j=0}^{\infty} \mathbb{P}(X_0 = j)\xi^j \Leftarrow \text{only } \mathbb{P}(X_0 = I) = 1 \\ &= \mathbb{P}(X_0 = I) * \xi^I = \xi^I \end{aligned}$$

(e)

$$\frac{\partial \Delta}{\partial t} = \sum_{j=0}^{\infty} p_j'(t)\xi^j$$

Substituting the $p_j'(t)$ we get from part (c) into above euqation.

(f)

$$\begin{aligned}\frac{\partial \Delta}{\partial \xi}(1,t) &= \frac{\partial}{\partial \xi}\sum_{j=0}^{\infty} p_j(t)\xi^j\Big|_{\xi=1} = \sum_{j=0}^{\infty} jp_j(t) \\ &= \sum_{j=0}^{\infty} j\,\mathbb{P}(X_t = j) = \mathbb{E}[X_t] \\ \Rightarrow \mathbb{E}[X_t] &= \frac{\partial \Delta}{\partial \xi}(1,t)\end{aligned}$$

(g)

$$\begin{aligned}\frac{\partial^2 \Delta}{\partial \xi^2}(1,t) &= \sum_{j=0}^{\infty} j(j-1)p_j(t)\xi^{j-2}\Big|_{\xi=1} \\ &= \sum_{j=0}^{\infty} j^2 p_j(t) - \sum_{j=0}^{\infty} jp_j(t) \\ &= \mathbb{E}[X_t^2] - \mathbb{E}[X_t] \Leftarrow Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2, \text{So we add } (\mathbb{E}[X])^2, \text{then subtract it} \\ &= \mathbb{E}[X_t^2] - (\mathbb{E}[X])^2 - \mathbb{E}[X_t] + (\mathbb{E}[X])^2 \\ &= Var(X_t) - \mathbb{E}[X_t] + (\mathbb{E}[X])^2 \Leftarrow \text{Substituting the result of (f)} \\ \Rightarrow Var(X_t) &= \frac{\partial^2 \Delta}{\partial \xi^2}(1,t) + \frac{\partial \Delta}{\partial \xi}(1,t) - (\frac{\partial \Delta}{\partial \xi}(1,t))^2\end{aligned}$$

(h)

$$\begin{aligned}\mathbb{E}[X_t] &= \frac{\partial \Delta}{\partial \xi}(1,t) = \frac{\rho}{\mu}(1 - e^{-\mu t}) + Ie^{-\mu t} \\ \lim_{t \to 0^+} \mathbb{E}[X_t] &= I\end{aligned}$$

It behaves resonably.

(i)

$$\begin{aligned}Var(X_t) &= \frac{\partial^2 \Delta}{\partial \xi^2}(1,t) + \frac{\partial \Delta}{\partial \xi}(1,t) - (\frac{\partial \Delta}{\partial \xi}(1,t))^2 \\ &= \frac{\rho^2}{\mu^2}(1 - e^{-\mu t})^2 + I(I-1)e^{-2\mu t} + 2I\frac{\rho}{\mu}e^{-\mu t}(1 - e^{-\mu t}) \\ &+ \frac{\rho}{\mu}(1 - e^{-\mu t}) + Ie^{-\mu t} - \left(\frac{\rho}{\mu}(1 - e^{-\mu t}) + Ie^{-\mu t}\right)^2\end{aligned}$$

(j)

$$\lim_{t \to 0^+} Var(X_t) = 0$$

Because in the begining of this process, all is determined. and it's the same with our prediction after calculating the expression.

(k) By taking the n^{th} derivative w.r.t. ξ of $\Delta(\xi, t)$. firstly, let all terms with $j < n$ equal to 0, also by setting $\xi = 0$, we making all terms with $j > n$ equal to 0. So that only terms with n left with

$$\frac{\partial^n \Delta}{\partial \xi^n}(0, t) = n! p_n(t)$$

So for $\mathbb{P}(X_t = 0) = p_0(t) = \Delta(0, t) = e^{-\frac{\rho}{\mu}(1-e^{-\mu t})}\left[1 - e^{-\mu t}\right]^I$

(l) Using the same equation derived from part (k), we get

$$\begin{aligned}\mathbb{P}(X_t = 1) = p_1(t) &= \frac{\partial \Delta}{\partial \xi}(0, t) \\ &= e^{-\frac{\rho}{\mu}(1-e^{-\mu t})}\left[1 - e^{-\mu t}\right]^I \left[\frac{\rho}{\mu}(1 - e^{-\mu t}) + I\frac{e^{-\mu t}}{1 - e^{-\mu t}}\right]\end{aligned}$$

(m)

$$\lim_{t\to\infty,\rho\to 0} \mathbb{E}[X_t] = \frac{\rho}{\mu}(1 - e^{-\mu t}) + Ie^{-\mu t} = \frac{\rho}{\mu} = 0$$

When $\rho \to 0$, it seems like a pure death process, in the end, the population will become zero.

(n)

$$\lim_{t\to\infty,\mu\to 0} \mathbb{E}[X_t] = \frac{\rho}{\mu}(1 - e^{-\mu t}) + Ie^{-\mu t} = \frac{\rho}{\mu} = \infty$$

It looks like a pure-birth process. so the expectation of population will be ∞ in a long time view.

(o) Using the reuslt of part (k),

$$\frac{\partial^n \Delta}{\partial \xi^n}(0, t) = n! p_n(t) \Rightarrow \lim_{t\to\infty} p_j(t) = \frac{\partial^n \Delta}{\partial \xi^n}(0, t)/j!$$

From the question we can get the $\lim_{t\to\infty} \Delta(\xi, t) = e^{\frac{\rho}{\mu}(\xi-1)}$. So

$$\lim_{t\to\infty} p_j(t) = e^{\frac{\rho}{\mu}(\xi-1)}\frac{(\frac{\rho}{\mu})^j}{j!}$$

(p) First, this chains is Irreducible, second, it's aperiodic, So it is Ergodic Markov chains. therefore, they eventually reach a unique stationary distribution, regredless of their initial state.

(q) By simply computing $\mathbf{P}^n$ by increasing n until the result is not change.

Problem 2

Ans:

(a) We known that this is a time-homogenerous Poisson process, the $N_t - N_s$ represents the number of events in time $(s, t]$. It's also a Poisson process with parameter $(t-s)\lambda$.

$$\begin{aligned} N_t - N_s = k \sim Poi((t-s)\lambda) &= e^{-(t-s)\lambda}\frac{((t-s)\lambda)^k}{k!} \\ \mathbb{E}[N_t - N_s] &= (t-s)\lambda \\ Var(N_t - N_s) &= (t-s)\lambda \end{aligned}$$

(b)

$$\begin{aligned} \mathbb{E}[(N_t - N_s)^2] &= Var(N_t - N_s) + (\mathbb{E}[N_t - N_s])^2 \\ &= (t-s)\lambda + (t-s)^2\lambda^2 \end{aligned}$$

(c)

$$\begin{aligned} \mathbb{E}[N_t^2] &= Var(N_t) + \mathbb{E}[N_t] \\ &= \lambda t + \lambda^2 t^2 \end{aligned}$$

(d)

$$\begin{aligned} \mathbb{E}[(N_t - N_s)^2] &= \mathbb{E}[N_t^2 + N_s^2 - 2N_tN_s] \\ &= \mathbb{E}[N_t^2] + \mathbb{E}][N_s^2] - 2\,\mathbb{E}[N_tN_s] = (t-s)\lambda + (t-s)^2\lambda^2 \\ \Rightarrow \mathbb{E}[N_tN_s] &= \frac{\mathbb{E}[N_t^2] + \mathbb{E}][N_s^2] - (t-s)\lambda + (t-s)^2\lambda^2}{2} \\ \mathbb{E}[N_s^2] &= \lambda s + \lambda^2 s^2 \end{aligned}$$

Finally, by solving the above equations, we get

$$\mathbb{E}[N_sN_t] = \lambda s + \lambda^2 st$$

(e)

$$\begin{aligned} \mathbb{E}[N_sN_t] &= \mathbb{E}[(N_t - N_s)(N_s - 0) + N_s^2] \\ &= \mathbb{E}[N_t - N_s]\,\mathbb{E}[N_s - N_0] + \mathbb{E}[N_s^2] \Leftarrow \text{we know that } N_t - N_s, N_s - N_0 \text{ are independent} \\ &= (t-s)\lambda * \lambda s + \lambda s + \lambda^2 s^2 \\ &= \lambda s + \lambda^2 st \end{aligned}$$

In the above process, we use the property that the increments of N_t are independent.

Problem 3

Ans:

(a)

$$p_{X_1}(n_1) = \mathbb{P}(X_1 = n_1) = \binom{n}{n_1}\left(\frac{1}{6}\right)^{n_1}\left(\frac{5}{6}\right)^{n-n_1}$$

(b)

$$\mathbb{E}[X_1] = np_1 = \frac{n}{6}$$

(c)

$$\begin{aligned}
p_{X_2|X_1}(k|n_1) = \mathbb{P}(X_2 = k|X_1 = n_1) &= \frac{\mathbb{P}(X_2 = k, X_1 = n_1)}{\mathbb{P}(X_1 = n_1)} \\
&= \frac{\binom{n}{n_1}\left(\frac{1}{6}\right)^{n_1}\binom{n-n_1}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{2}{3}\right)^{n-n_1-k}}{\binom{n}{n_1}\left(\frac{1}{6}\right)^{n_1}\left(\frac{5}{6}\right)^{n-n_1}} \\
&= \frac{\binom{n-n_1}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{2}{3}\right)^{n-n_1-k} * \left(\frac{5}{6}\right)^{-k}}{\left(\frac{5}{6}\right)^{n-n_1} * \left(\frac{5}{6}\right)^{-k}} \\
&= \binom{n-n_1}{k}\left(\frac{1}{5}\right)^{k}\left(\frac{4}{5}\right)^{n-n_1-k}
\end{aligned}$$

(d) From (c) we know that $p_{X_2|X_1}$ is a binomial distribution with parameter $\frac{1}{5}$. So the $\mathbb{E}[X_2|X_1] = np = (n - n_1)\frac{1}{5}$

(e)

$$\begin{aligned}
\mathbb{E}[X_2] &= \mathbb{E}[\mathbb{E}[X_2|X_1]] \\
&= \mathbb{E}[\frac{1}{5}(n - n_1)] \\
&= \mathbb{E}[\frac{n}{5}] - \frac{1}{5}\mathbb{E}[X_1] \\
&= \frac{n}{5} - \frac{n}{30} \\
&= \frac{n}{6}
\end{aligned}$$

It's obvious that the result should be the same as $\mathbb{E}[X_1] = \frac{n}{6}$