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MATH 5763 — Fall 2015 **Final Exam** Due: 12/17/2015

Problem 1

Ans:

(a) It similar to how many steps you need to across a length of t road. if your first step length is $x_1 > t$, then you only need 1 step, so the $\mathbb{E}[N(t)|X_1 = x_1 > t] = 1$. if your first step is $x_1 \leq t$, then you need 1+$\mathbb{E}[t - x_1]$, because each step is independent.

(b)

$$\begin{aligned} M(t) = \mathbb{E}[N(t)] &= \int_0^1 \mathbb{E}[N(t)|X_1]p(x_1)dx_1 \\ &= \int_0^t 1 + M(t - x_1)dx_1 + \int_t^1 1dx_1 \\ &= t + \int_0^t M(t - x_1)dx_1 + 1 - t \\ &= 1 + \int_0^t M(t - x_1)dx_1 \end{aligned}$$

(c)

$$\begin{aligned} M(t) &= 1 + \int_0^t M(t - x_1)dx_1 \\ &= 1 + \int_t^0 -M(t - x_1)d(t - x_1) \\ &= 1 + \int_t^0 -M(y)dy \\ &= 1 + \int_0^t M(y)dy \end{aligned}$$

Differentiate both sides of the above equation, we got $M'(t) = M(t)$, so $M(t) = Ce^t$. For $t = 0$, it also like $t < x_1$. the $M(0) = 1$. So we can solve the initial problem for $M(t)$ as $M(t) = e^t$

(d)

$$F = \frac{F}{S} + \frac{1}{S} \Longrightarrow F = \frac{1}{S - 1}$$

From the Laplace transform table, we know that $M(t) = e^t$

Problem 2

Ans:

(a) The sum of the probability for $p_{ii}, p_{i,i_1}, p_{i,i+1}$ is 1. so the $p_{ii} = 1 - p_{i,i+1} - p_{i,i_1}$;
when $i \leq c$, which means that all of them are will be serving at the same time. if $i > c$, only c customers will be served.

(b)

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & \\ 0 & \cdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & i\mu & -(\lambda+i\mu) & \rho \end{pmatrix}$$

(c) The system of equations $\pi G = 0$ becomes

$$-\lambda\pi_0 + \mu\pi_1 = 0,$$
$$\cdots$$
$$\lambda\pi_{i-1} - (\lambda + i\mu)\pi_i + (i+1)\mu\pi_{i+1} = 0$$

(d)

$$\begin{aligned} N_Q &= \sum_{n=c}^{\infty}(n-c)\pi_n \\ &= \pi_0 \frac{(c\rho)^c}{c!}\sum_{n=c}^{\infty}(n-c)\rho^{n-c} \\ &= \frac{\pi_c\rho}{(1-\rho)^2} \end{aligned}$$

(e)

$$\begin{aligned} P_Q &= \sum_{n=c}^{\infty}\pi_n \\ &= \pi_0 \frac{(c\rho)^c}{c!}\sum_{n=c}^{\infty}\rho^{n-c} \\ &= \frac{\pi_c}{1-\rho} \end{aligned}$$

(f)

$$P[q>0] = \sum_{i=c+1}^{\infty} = \frac{\pi_c}{1-\rho} - \pi_c = \frac{\pi_c\rho}{1-\rho}$$

(g)

$$\begin{aligned} E[q] &= P[q=0]\,\mathbb{E}[q|q=0] + P[q>0]\,\mathbb{E}[q|q>0] \\ &= (1-P[q>0]) * 0 + P[q>0] * \mathbb{E}[q|q>0] = \frac{\pi_c\rho}{(1-\rho)^2} \\ &\Rightarrow \mathbb{E}[q|q>0] = \frac{1}{1-\rho} \end{aligned}$$

(h) When the system reached stationary state, the length of queuing is $L = \lambda * W$, where W is the average time one spent on the system. so the mean number of customers that are served in 1 unit of time is λ.

(i)

$$W = \frac{N_Q}{\lambda} = \frac{\pi_c}{c\mu(1-\rho)^2}$$

Problem 3

Ans:

(a)

$$\begin{aligned} dX_t &= -kX_t dt + \alpha dB_t \\ &\Rightarrow e^{kt} dX_t = -e^{kt} k X_t dt + e^{kt} \alpha dB_t \end{aligned}$$

(b)

$$\begin{aligned} d(e^{kt}X_t) &= ke^{kt}X_t d_t + e^{kt} dX_t \text{ from (a) we know } e^{kt} dX_t \\ &= \alpha e^{kt} dB_t \end{aligned}$$

(c) From above (b) we know that $e^{kt}X_t - X_0 = \int_0^t \alpha e^{kt} dB_t$. So we can get that:

$$\begin{aligned} e^{kt}X_t &= X_0 + \int_0^t \alpha e^{kt} dB_t \\ &\Rightarrow X_t = e^{-kt}\left(X_0 + \int_0^t \alpha e^{kt} dB_t\right) \end{aligned}$$

(d)

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}[X_0] * e^{-kt} + \mathbb{E}\left[\alpha e^{-kt} \int_0^t e^{ks} dB_s\right] \\ &= e^{-kt}\,\mathbb{E}[X_0] \end{aligned}$$

(e)

$$\begin{aligned} \mathbb{E}[X_t^2] &= \mathbb{E}[X_0^2] * e^{-2kt} + \mathbb{E}\left[\left(\alpha e^{-kt} \int_0^t e^{ks} dB_s\right)^2\right] \\ &= e^{-2kt}\,\mathbb{E}[X_0^2] + \alpha^2 e^{-2kt}\,\mathbb{E}\left[\frac{e^{2ks}}{2k}\Big|_0^t\right] \\ &= e^{-2kt}\,\mathbb{E}[X_0^2] + \alpha^2 e^{-2kt}\left(\frac{e^{2kt}-1}{2k}\right) \\ varX_t &= \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 \\ &= \frac{\alpha^2}{2k} + \left(varX_0 - \frac{\alpha^2}{2k}\right)e^{-2kt} \end{aligned}$$

(f) From equation (3) we know that $f(t, X_t) = -kX_t, g(t, X_t) = \alpha$, so the Fokker-Planck equation for equation (3) is:

$$\begin{aligned}\frac{\partial \rho(x,t|x_0,t_0)}{\partial t} &= -\frac{\partial}{\partial x}[-kX_t\rho(x,t|x_0,t_0)] + \frac{\partial^2}{2\partial x^2}[\alpha^2\rho(x,t|x_0,t_0)] \\ &= \frac{\partial}{\partial x}[kX_t\rho(x,t|x_0,t_0)] + \frac{\partial^2}{2\partial x^2}[\alpha^2\rho(x,t|x_0,t_0)]\end{aligned}$$

The $\rho(x, t|x_0, t_0)$ is differentiable.

(g) In the last equality, we use Fokker-Planck equation.

(h)

$$\lim_{t\to t_0^+} M(\theta, t|X_{t_0} = x_0) = e^{x_0\theta}$$

(i)

$$\begin{aligned}\mathbb{E}[X_t|X_{t_0} = x_0] &= M'(0,t) = x_0 e^{-k(t-t_0)} \text{ partially derivative w.r.t.}\theta \\ var(X_t|X_{t_0} = x_0) &= \mathbb{E}[X_t^2|X_{t_0} = x_0] - (\mathbb{E}[X_t|X_{t_0}])^2 \\ &= M''(0,\theta) - x_0^2 e^{-2k(t-t_0)} \\ &= \frac{\alpha^2}{2k} + \left(x_0^2 - \frac{\alpha^2}{2k}\right)e^{-2k(t-t_0)} - x_0^2 e^{-2k(t-t_0)}\end{aligned}$$

(j) The X_t is a Normal distribution with parameters $N(x_0 e^{-k(t-t_0)}, \frac{\alpha^2}{2k}(1 - e^{-2k(t-t_0)})$.

(k) $t \to t_0^+$ it is in the begining state of x_0;
$t \to \infty$, then the $\mathbb{E}[X_t|X_{t_0} = x_0] = 0$ means that it will stop;
$\alpha \to 0^+$, then $var(X_t|X_{t_0} = x_0) = 0$;
$k \to 0^+$, means no frictional force, the state will always x_0.

(l)

$$\begin{aligned}\rho = (x,t|x_0,t_0) &:= f_{X_t|X_{t_0}}(x|x_0) \\ &\Rightarrow f_{X_t}(x) = \int_0^t f_{X_{t_0}}(x) * \rho dx \\ \mathbb{E}[X_t] &= \int_0^t X_t f_{X_t}(x)dx \\ varX_t &= \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2\end{aligned}$$