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ISE 5643 — Spring 2015 **Final Exam** Due: Apr. 13

Please show all of your work so that it is clear how you arrived at your solution and so that you may receive partial credit where applicable. The entire exam is worth 100 points with the point values for individual problems indicated in parenthesis after each problem.

Any form of academic dishonesty will be dealt with severely according to the limits of the code.

Problem 1

Suppose the grade point average (GPA) for a student can be accurately predicted from the students GMAT (Graduate Management Admissions Test) score. More specifically, suppose that the ith student observed has a GPA of y_i and a GMAT score of x_i. How can we use the least squares methods to estimate a hypothesized relation of the form $y_i = a + bx_i$?

Ans: Let $\hat{a}$ be our estimate of a and $\hat{b}$ our estimate of b. Given that for students $i = 1, 2, ..., n$ we have observed $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$, $\hat{e}_i = y_i - (\hat{a} + \hat{b}x_i)$ is our error in estimating the GPA of student i. The least squares method chooses $\hat{a}$ and $\hat{b}$ to minimize

$$\min f(a,b) = \sum_i (y_i - \hat{y}_i)^2$$

Since

$$\begin{aligned} \frac{\partial f}{\partial a} &= -2\sum_{i=1}^{i=n}(y_i - a - bx_i) \\ \frac{\partial f}{\partial b} &= -2\sum_{i=1}^{i=n}(y_i - a - bx_i)x_i \end{aligned}$$

setting $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = 0$, we get the point $(\hat{a}, \hat{b})$. Then proving that the Hessian for $f(a,b)$ $H = \begin{pmatrix} 2n & 2\sum_{i=1}^{i=n} x_i \\ 2\sum_{i=1}^{i=n} x_i & 2\sum_{i=1}^{i=n} x_i^2 \end{pmatrix} > 0.$

From equation:

$$n\sum_{i=1}^{i=n} x_i^2 \geq \left(\sum_{i=1}^{i=n} x_i\right)^2$$

with equality holding if and only if $x_1 = x_2 = ... = x_n$. Thus, if at least two of the x_i's are different, which implies that $(\hat{a}, \hat{b})$ will be a local minimum. It does not depend on the values of a and b.

Problem 2

Consider the NLP problem:

$$\begin{aligned}
&\min f(x) = -x_2 \\
&\text{s.t.} \\
&g_1(x) = 49 - x_1^2 - 3x_2^2 \geq 0 \\
&g_2(x) = 28 - (x_1 - 1)^2 + x_1x_2 - 2x_2^2 \geq 0 \\
&g_3(x) = 22 - 2x_1 - 5x_2 \geq 0
\end{aligned}$$

(a) Write down the Kuhn-Tucker conditions.

Ans:

$$L(x_1, x_2, u_1, u_2, u_3) = -x_2 - u_1g_1 - u_2g_2 - u_2g_3$$

KTCs:

$$\begin{aligned}
2u_1x_1 + 2u_2x_1 - 2u_2 - u_2x_2 + 2u_3 &= 0 \\
-1 + 6u_1x_2 - x_1u_2 + 4u_2x_2 + 5u_3 &= 0 \\
g_1(x) &\geq 0 \\
g_2(x) &\geq 0 \\
g_3(x) &\geq 0 \\
u_1g_1(x) &= 0 \\
u_2g_2(x) &= 0 \\
u_3g_3(x) &= 0 \\
u_1, u_2, u_3 &\geq 0
\end{aligned}$$

(b) Show that the Kuhn-Tucker conditions are both necessary and sufficient. You must justify your conclusions.

Ans: First proving KTCs are necessary: To verify that the $g(x)$ function is concave function, and existing a point satisfy the equations.

$$H_{g_1}(x) = \begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix}, H_{g_2}(x) = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix} H_{g_3}(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then proving KTCs are sufficient:The objective function is convex function.

(c) Using (a) and (b) prove that the solution $x_1 = 1, x_2 = 4$ is optimal for the NLP. Justify your steps.

Ans: Substituting $x_1 = 1, x_2 = 4$ to KTCs, we get $u_1 = \frac{2}{63}, u_2 = \frac{1}{63}, u_3 = 0$, satisfying the conditions, so it's sufficient, at the same time it is necessary condition, so we can get that the solution is optimal for the NLP.

Problem 3

Consider the following problem:

$$\max x_1^2 + 4x_1x_2 + x_2^2$$
s.t.
$$x_1^2 + x_2^2 = 1$$

(a) Using the KKT conditions, find an optimal solution to the problem.

Ans: Maximizing the object function $x_1^2 + 4x_1x_2 + x_2^2$ equals to minimizing the $-(x_1^2 + 4x_1x_2 + x_2^2)$.

$$L(x_1, x_2, v) = -(x_1^2 + 4x_1x_2 + x_2^2) - v(x_1^2 + x_2^2 - 1)$$

KTCs:

$$\begin{aligned} -2x_1 - 4x_2 - 2vx_1 &= 0 \\ -2x_2 - 4x_1 - 2vx_2 &= 0 \\ v(x_1^2 + x_2^2 - 1) &= 0 \end{aligned}$$

Using MATLAB:

```
[a,x,y]=solve('-2*x-4*y-2*a*x=0','-2*y-4*x-2*a*y=0','a*(x^2+y^2-1)=0')
```

we get

$$x_1 = 0, x_2 = 0, v = 0,$$
$$x_1 = -\frac{\sqrt{2}}{2}, x_2 = \frac{\sqrt{2}}{2}, v = 1,$$
$$x_1 = \frac{\sqrt{2}}{2}, x_2 = \frac{\sqrt{2}}{2}, v = -3,$$
$$x_1 = \frac{\sqrt{2}}{2}, x_2 = -\frac{\sqrt{2}}{2}, v = 1,$$
$$x_1 = -\frac{\sqrt{2}}{2}, x_2 = -\frac{\sqrt{2}}{2}, v = -3,$$

Finally, only $x_1 = x_2 = \frac{\sqrt{2}}{2}$,or $x_1 = x_2 = -\frac{\sqrt{2}}{2}$ at $v = -3$, which satisfy the conditions, which maximize the object function.

(b) Test for the second-order optimality conditions.

Ans: For $v = -3$, we compute the $H_f = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$ with respect to $v = -3$. which is a PSD matrix.

(c) Does the problem have a unique optimal solution?

Ans: No.

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Problem 4

Consider the following problem:

$$\begin{aligned}
&\max 2x_1 + 3x_2 \\
&\text{s.t.} \\
&x_1 + x_2 \leq 8 \\
&-x_1 + 2x_2 \leq 4 \\
&x_1, x_2 \geq 0
\end{aligned}$$

(a) Write down the KKT conditions.

Ans:

$$\begin{aligned}
L(x_1, x_2, u_1, u_2, u_3, u_4) &= 2x_1 + 3x_2 + u_1(x_1 + x_2 - 8) + u_2(-x_1 + 2x_2 - 4) \\
&+ u_3(-x_1) + u_4(-x_2)
\end{aligned}$$

KTCs:

$$\begin{aligned}
2 + u_1 - u_2 - u_3 &= 0 \\
3 + u_1 + 2u_2 - u_4 &= 0 \\
x_1 + x_2 - 8 &\leq 0 \\
-x_1 + 2x_2 - 4 &\leq 0 \\
u_1(x_1 + x_2 - 8) &= 0 \\
u_2(-x_1 + 2x_2 - 4) &= 0 \\
u_3(-x_1) &= 0 \\
u_4(-x_2) &= 0 \\
u_1, u_2, u_3, u_4 &\leq 0
\end{aligned}$$

(b) For each extreme point, verify whether or not the KKT conditions are true, algebraically and geometrically. From this, find the optimal solution.

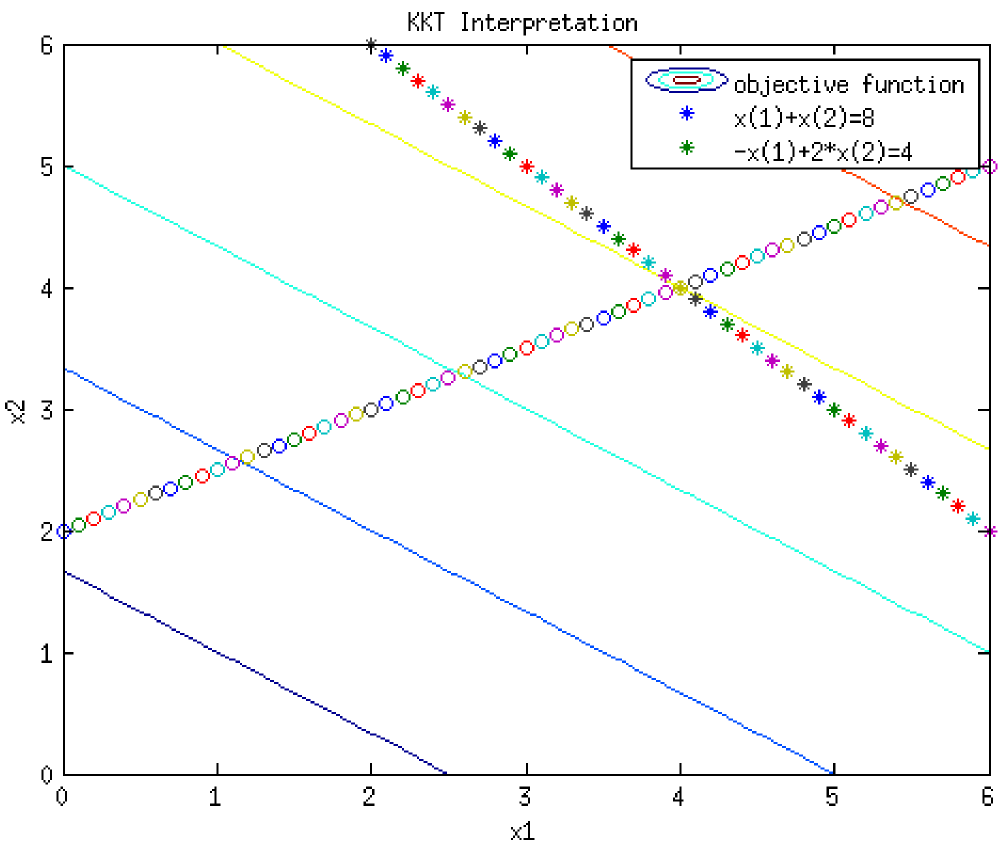
Ans: Using MATLAB:

```
[a,b,c,d,x,y]=solve('2+a-b-c=0','3+a+2*b-d=0','a*(x+y-8)=0
    ','b*(-x+2*y-4)=0','-x*c=0','-y*d=0')
```

we get

$$
\begin{aligned}
u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 3, x_1 = 0, x_2 = 0 \\
u_1 = 0, u_2 = 2, u_3 = 0, u_4 = 7, x_1 = -4, x_2 = 0 \\
u_1 = -2, u_2 = 0, u_3 = 0, u_4 = 1, x_1 = 8, x_2 = 0 \\
u_1 = -3, u_2 = 0, u_3 = -1, u_4 = 0, x_1 = 0, x_2 = 8 \\
u_1 = 0, u_2 = -\frac{3}{2}, u_3 = \frac{7}{2}, u_4 = 0, x_1 = 0, x_2 = 2 \\
u_1 = -\frac{7}{3}, u_2 = -\frac{1}{3}, u_3 = 0, u_4 = 0, x_1 = 4, x_2 = 4
\end{aligned}
$$

From algebraically we substituting the u_i, x_i to the conditions, we get only one point that satisfy the conditions, Drawing the conditions and objective as below:



Thus, the optimal solution is $x_1 = x_2 = 4$, the value is $f(4, 4) = 20$

Problem 5

Consider the following problem:

$$\begin{aligned}
&\min(x_1-\frac{9}{4})^2+(x_2-2)^2\\
&\text{s.t.}\\
&x_2-x_1^2\geq 0\\
&x_1+x_2\leq 6\\
&x_1,x_2\geq 0
\end{aligned}$$

(a) Write down the Kuhn-Tucker optimality conditions and verify that these conditions are true at the point $\bar{x}=(\frac{3}{2},\frac{9}{4})$.

Ans:

$$\begin{aligned}
L(x_1,x_2,u_1,u_2,u_3,u_4) &= (x_1-\frac{9}{4})^2+(x_2-2)^2-u_1(x_2-x_1^2)\\
&- u_2(6-x_1-x_2)-u_3x_1-u_4x_2
\end{aligned}$$

KTCs:

$$\begin{aligned}
2(x_1-\frac{9}{4})+2u_1x_1+u_2-u_3 &= 0\\
2(x_2-2)-u_1+u_2-u_4 &= 0\\
x_2-x_1^2 &\geq 0\\
6-x_1-x_2 &\geq 0\\
x_1,x_2 &\geq 0\\
u_1(x_2-x_1^2) &= 0\\
u_2(6-x_1-x_2) &= 0\\
u_3x_1 &= 0\\
u_4x_2 &= 0\\
u_1,u_2,u_3,u_4 &\geq 0
\end{aligned}$$

Using MATLAB:

```
[a,b,c,d,x,y]=solve('2*(x-2.25)+2*a*x+b-c=0','2*(y-2)-a+b-
    d=0','a*(y-x^2)=0','b*(6-x-y)=0','x*c=0','y*d=0')
```

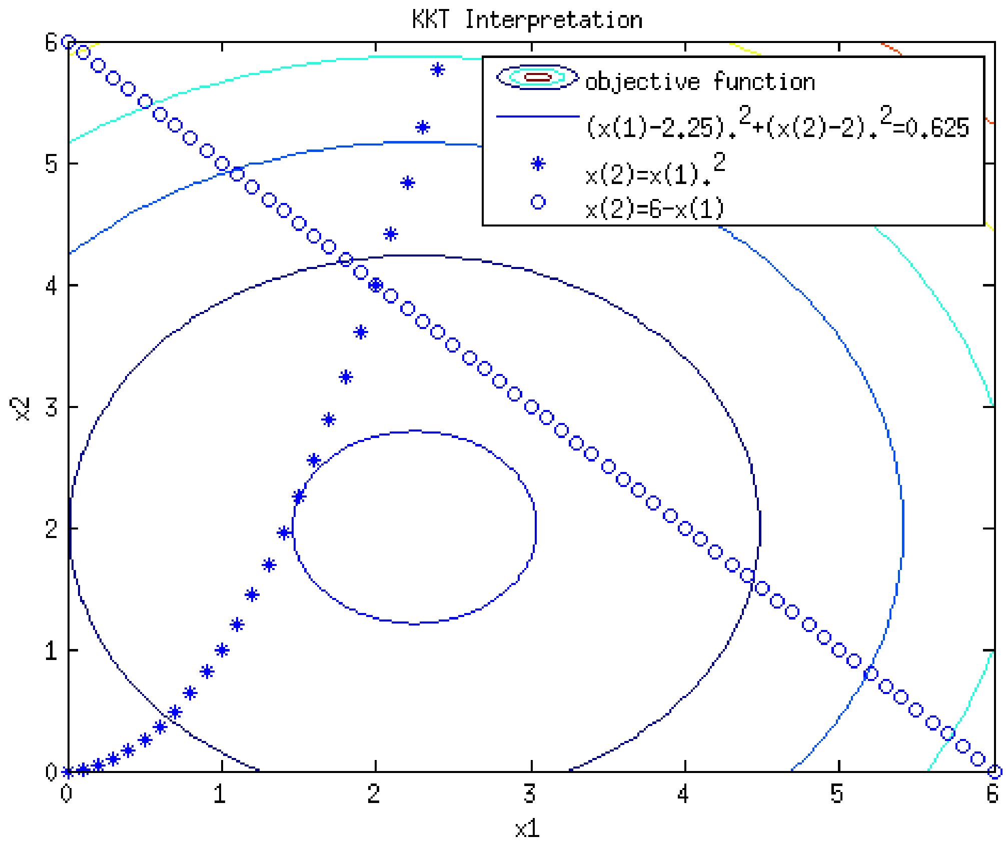
we get

$$u_1 = -1 * z - 4, u_2 = 0, u_3 = -4.5, u_4 = z, x_1 = 0, x_2 = 0$$
$$u_1 = 0, u_2 = 0, u_3 = -4.5, u_4 = -4, x_1 = 0, x_2 = 0$$
$$u_1 = 0, u_2 = 0, u_3 = -4.5, u_4 = 0, x_1 = 0, x_2 = 2$$
$$u_1 = -4, u_2 = 0, u_3 = -4.5, u_4 = 0, x_1 = 0, x_2 = 0$$
$$u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0, x_1 = 2.25, x_2 = 2$$
$$u_1 = 0, u_2 = 0, u_3 = 0, u_4 = -4, x_1 = 2.25, x_2 = 0$$
$$u_1 = 0, u_2 = -8, u_3 = -12.5, u_4 = 0, x_1 = 0, x_2 = 6$$
$$u_1 = 0.5, u_2 = 0, u_3 = 0, u_4 = 0, x_1 = 1.5, x_2 = 2.25$$
$$u_1 = 0, u_2 = -7.5, u_3 = 0, u_4 = -11.5, x_1 = 6, x_2 = 0$$
$$u_1 = 0, u_2 = -1.75, u_3 = 0, u_4 = 0, x_1 = 3.125, x_2 = 2.875$$
$$u_1 = -4.9, u_2 = -18.9, u_3 = 0, u_4 = 0, x_1 = -3, x_2 = 9$$
$$u_1 = 0.9, u_2 = -2.1, u_3 = 0, u_4 = 0, x_1 = 2, x_2 = 4$$

Verifying all the possibilities, we get that when $u_1 = \frac{1}{2}, u_2 = u_3 = u_4 = 0, x_1 = 1.5, x_2 = 2.25$, these variables satisfy the above KTCs.

(b) Interpret the KKT conditions at $\bar{x}$ graphically.

Ans: We draw the object function below, also we draw the two constrain condition as ***** and **o**. finally, we draw a circle, with equation $(x_1 - 2.25)^2 + (x_2 - 2)^2 = 0.625$. the problem can be seen as computing the minimal distance between feasible area and a circle with a center $(\frac{9}{4}, 2)$.



(c) Show that $\bar{x}$ is indeed the unique global optimal solution.

Ans: As the figure above, the position $\bar{x} = (\frac{3}{2}, \frac{9}{4})$ is exactly the point of tangency, thus it is the optimal solutions. Also we can computing the Hessian for object function $H_f = \begin{pmatrix} 2+2u_1 & 0 \\ 0 & 2 \end{pmatrix}$. Considering $u >= 0$, we know H is PD, at the same time, we can prove $g_i(x)$ are concave functions, So the point satisfy the KTCs is also the optimal solution.