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ISE 5643 — Spring 2015 **Exam 2** Due: Apr. 3

Please show all of your work so that it is clear how you arrived at your solution and so that you may receive partial credit where applicable. The entire exam is worth 100 points with the point values for individual problems indicated in parenthesis after each problem.

Any form of academic dishonesty will be dealt with severely according to the limits of the code.

Problem 1

Suppose we are minimizing a function $f(x)$ using a gradient algorithm. At some point $x = \bar{x}$, we have **(16 pts)**

$$\bigtriangledown f(\bar{x}) = (1, -1) \textit{ and } H_f(\bar{x}) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

Calculate the search direction at $\bar{x}$ using the following methods:

(a) Cauchy

(b) Newton

(c) Marquardt with $\lambda = 5$

Prove whether or not the search directions obtained in parts (a), (b), and (c) are descent directions.

Ans:

(a) Cauchy

$$d = -\frac{\bigtriangledown f(\bar{x})}{|| \bigtriangledown f(\bar{x})||} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

Because $- \bigtriangledown f(\bar{x}) \cdot d > 0$, So it is descent direction.

(b) Newton

$$d = -\frac{\bigtriangledown f(\bar{x})}{H_f(\bar{x})} = (-1, 0)$$

Because $- \bigtriangledown f(\bar{x}) \cdot d > 0$, So it is descent direction.

(c) Marquardt with $\lambda = 5$

$$d = -\frac{\bigtriangledown f(\bar{x})}{H_f(\bar{x}) + \lambda I} = (-0.1379, 0.1724)$$

Because $- \bigtriangledown f(\bar{x}) \cdot d > 0$, So it is descent direction.

Problem 2

Consider the NLP problem: **(20 pts)**

$$\min z = (x_1 - 1)^2 + (x_2 - 1)^2$$
s.t.
$$-x_1 + x_2 = 1$$
$$x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

(a) Write down the Kuhn-Tucker conditions.

Ans:

$$L(x_1, x_2, u, v) = (x_1 - 1)^2 + (x_2 - 1)^2 - u_1 x_1 - u_2 x_2 - u_3(2 - x_1 - x_2) - v(1 + x_1 - x_2)$$

KTCs:

$$\begin{aligned} 2(x_1 - 1) - u_1 + u_3 - v &= 0 \\ 2(x_2 - 1) - u_2 + u_3 + v &= 0 \\ x_1, x_2 &\geq 0 \\ 2 - x_1 - x_2 &\geq 0 \\ 1 + x_1 - x_2 &= 0 \\ u_1 x_1 &= 0 \\ u_2 x_2 &= 0 \\ u_3(2 - x_1 - x_2) &= 0 \\ u_1, u_2, u_3 &\geq 0 \end{aligned}$$

(b) Show that the Kuhn-Tucker conditions are both necessary and sufficient. You must justify your conclusions.

If the objective function is convex function, and the $g(x)$ function is concave function, and the $h(x)$ are independent linear, than the KTCs conditions are necessary and sufficient.

Problem 3

A company is planning to spend \$10,000 on advertising. It costs \$3,000 per minute to advertise on television and \$1,000 per minute to advertise on radio. If the firm buys x minutes of television advertising and y minutes of radio advertising, then its revenue in thousands of dollars is given by $f(x, y) = -2x^2 - y^2 + xy + 8x + 3y$. How can the firm maximize its revenue? .**(18 pts)**

Ans: Constructing the model of this problem as below:

$\max f(x,y) = \min -f(x,y) = 2x^2 + y^2 - xy - 8x - 3y$
s.t.
$3000x + 1000y \leq 10000$
$x, y \geq 0$

(1) First, Constructing the Lagrange function:

$$L(x, y, u_1, u_2, u_3) = 2x^2 + y^2 - xy - 8x - 3y - u_1(10 - 3 * x - y) - u_2 x - u_3 y$$

(2) Setting $\partial L/\partial x = \partial L/\partial y = 0$,gives

$$x = \frac{19 - 7u_1 + 2u_2 + u_3}{7}, y = \frac{20 - 7u_1 + u_2 + 4u_3}{7}$$

(3) Further, Substituting the above equations to the constraint condition.

$$u_1 * (10 - 3 * (19 - 7 * u_1 + 2 * u_2 + u_3)/7 - (20 - 7 * u_1 + u_2 + 4 * u_3)/7) = 0$$
$$u_2 * (19 - 7 * u_1 + 2 * u_2 + u_3)/7 = 0$$
$$u_3 * (20 - 7 * u_1 + u_2 + 4 * u_3)/7 = 0$$

(4) we get that $u_1 = 1/4, u_2 = 0, u_3 = 0; x = 69/28, y = 73/28$. thus, the $f(x, y) = 15.018$.

Problem 4

Consider the NLP problem below: **(18 pts)**

$\min f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 2)^2$
s.t.
$(x_1, x_2) \in R^2$

(a) Write the first-order optimality conditions. Are those conditions also sufficient for optimality? Why?

The first-order optimality conditions are $\bigtriangledown f(x) = 0$

$$\frac{\partial f}{\partial x_1} = 2(x_1 - 3) = 0 \Longrightarrow x_1 = 3$$
$$\frac{\partial f}{\partial x_2} = 2(x_2 - 2) = 0 \Longrightarrow x_2 = 2$$

Given point $(3, 2)'$, the $H_f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. it is PD matrix, so those conditions are sufficient for optimality.

(b) Is $\bar{x} = (0, -1)'$ an optimal solution? If not, use the method of steepest descent to identify a direction **d** along which the function would decrease.

$\bar{x} = (0, -1)'$ is not an optimal solution, because it did not satisfy the first-order optimality conditions.
The direction $\mathbf{d} = -\bigtriangledown f(\bar{x}) = (6, 6)'$, it is descent direction considering $-\bigtriangledown f(\bar{x}) \cdot \mathbf{d} > 0$.

(c) Using the method of steepest descent, minimize the function starting from (1, 1).

To begin, calculate the

$$\bigtriangledown f(x) = (2(x_1 - 3), 2(x_2 - 2)) = (-4, -2).$$

Then, $x^{(k+1)} = x^{(k)} - \alpha^{(k)} \bigtriangledown f(x^{(k)})$. So we need to calculate the α firstly, $\phi(\alpha) = f(x^k - \alpha^{(k)} \bigtriangledown f(x^k))$ is minimum along the $f(x^k - \alpha \bigtriangledown f(x^k))$.

$$\phi'(\alpha) = -\bigtriangledown f(x^0 - \alpha \bigtriangledown f(x^0)) \cdot \bigtriangledown f(x^0) = 0$$

Solving the above equation, we may get $\alpha = 0.5$; So the next point is $(1, 1)' - 0.5(-4, -2) = (3, 2)'$. and it is also the optimal point of this function.

Problem 5

What is the descent property? Why is it necessary? Cite one optimization method that does have and one that does not have this property.**(10 pts)**

Ans: Descent property means the method will iterate with

$$f(x^{(k+1)}) \leq f(x^{(k)})$$

It assure that the function will toward the minimum. Cauchy's Method is kind of these methods. Newton's method dose not have this property.

Problem 6

Determine whether or not the following is convex, concave or neither:**(8 pts)**

$$f(x_1, x_2) = (x_1 x_2)^2$$

The Hessian matrix of this function is $H_f = \begin{pmatrix} 2x_2^2 & 4x_1x_2 \\ 4x_1x_2 & 2x_1^2 \end{pmatrix}$ Due to the indefinite of the Hessian matrix, this function is neither convex nor concave.

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Problem 7

Answer True (T) or False (F). No explanation is necessary. **(10 pts)**

(a) Marquardts method does not require a line search.

T

(b) In constrained optimization, the optimal solution must be a Kuhn-Tucker point.

F

(c) An optimal solution to a nonlinear program cannot be in the interior of the feasible region.

F

(d) Kuhn-Tucker Necessity theorem can be used to prove that a given point is optimum.

F

(e) At a stationary point $\bar{x}$ of a multivariable function $f(x)$, $H_f(\bar{x})$ is PD. Then we conclude that $\bar{x}$ is a global minimum.

F