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ISE 5643 — Spring 2015 **Exam 1** Due: Mar. 2

Please show all of your work so that it is clear how you arrived at your solution and so that you may receive partial credit where applicable. The entire exam is worth 100 points with the point values for individual problems indicated in parenthesis after each problem.

Any form of academic dishonesty will be dealt with severely according to the limits of the code.

Problem 1

Consider the following unconstrained optimization problem.**(12 pts)**

$$\min f(x_1, x_2) = x_1^3 - 12x_1x_2 + 8x_2^3$$

a. Write the first-order optimality conditions. Are those conditions also sufficient for optimality? Why?

The first-order optimality condition is the first-derivative of $f' = 0$, we get the following equations:

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 3x_1^2 - 12x_2 \\ \frac{\partial f}{\partial x_2} &= -12x_1 + 24x_2^2 \end{aligned}$$

Solving the above equations, we get two points $(0, 0), (2, 1)$. But we cannot decide whether it's the minimal points, because it can be also be maximal point, depending on the property of the objective function.

b. Identify the critical points of f.

By constructing the Hessian matrix of objective function.

$$H_f = \begin{pmatrix} 6x_1 & -12 \\ -12 & 48x_2 \end{pmatrix}$$

For point $(0, 0)$, $D_1 = 0, D_2 > 0$, it's a Saddle point.
For point $(2, 1)$, $D_1 = 12 > 0, D_2 = 432 > 0$, it's a minimal point .

c. Is $\bar{x} = (0, 0)'$ an optimal solution? If not, explain why.

So $\bar{x} = (0, 0)'$ is not an optimal solution. the point $(2, 1)$ is the optimal solution, the reason is it's a Saddle point.

Problem 2

Suppose the demand $d_1, ..., d_n$ for a certain product over n periods is known. The demand during period j can be met from the production x_j during the period or from the warehouse stock. Any excess production can be stored at the warehouse. However, the warehouse has capacity K, and it would cost \$$c$ to carry over one unit from one period to another. The cost of production during period j is given by $f(x_j)$ for $j = 1, ..., n$. If the initial inventory is I_0, formulate the production scheduling problem as a nonlinear program.**(15 pts)**

a. Define your variables clearly. Identify the state variables and the control variables.
$i \in 1, ..., n$

x_i denoted the production of each period i;
I_i denoted the inventory of each period i;
d_i denoted the demand of each period i;
C is the cost function.

b. The constraints:

- $x_i - I_i + I_{i-1} = d_i$ where $i \in 1, ..., n$;
- $I_i < K$ the capacity constraints where $i \in 1, ..., n$;
- $x_i, I_i \geq 0$ nonnegative constraints where $i \in 1, ..., n$;

c. The objective function:

$$\begin{aligned} C &= \min \sum_{i=1}^{n} (f(x_i) + c * I_i) \\ &= \min \sum_{i=1}^{n} (f(x_i) + c * (x_i - d_i + I_{i-1})) \end{aligned}$$

where I_0 is known, and $i \in 1, ..., n$.

Problem 3

A monopolist producing a single product has two types of customers. If q_1 units are produced for customer 1, then customer 1 is willing to pay a price of $70 - 4q_1$ dollars. If q_2 units are produced for customer 2, then customer 2 is willing to pay a price of $150 - 15q_2$ dollars. For $q > 0$, the cost of manufacturing q units is $100 + 15q$ dollars. To maximize profit, how much should the monopolist sell to each customer? Explain why the solution is optimal.**(15 pts)**

The objective function is:

$$P = q_1(70 - 4q_1) + q_2(150 - 15q_2) - (100 + 15q) \; where \; q = q_1 + q_2$$

First find the stationary points of this function:

$$\bigtriangledown f(x)^T = (-8q_1 + 55, -30q_2 + 135) = (0, 0)$$

Then, we got that $q_1 = 6.875, q_2 = 4.5$. After computing the Hessian matrix of this function:

$$H_f = \begin{pmatrix} -8 & 0 \\ 0 & -30 \end{pmatrix}$$

We know that it's a concave function, so the stationary point also is a global maximal point.

Problem 4

Let $f : E_n \to E_1$ be a differentiable function. If f is twice differentiable at $\bar{x}$, then the quadratic approximation of f at $\bar{x}$ is given by **(10 pts)**

$$f(\bar{x}) + \bigtriangledown f(\bar{x})^T(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T H(\bar{x})(x - \bar{x})$$

Let $f(x_1, x_2) = e^{x_1^2+x_2^2} - 5x_1 + 10x_2$. Given the quadratic approximation of f at $(0, 1)$, is this approximation convex, concave, or neither? Why?

First, computing the $\bigtriangledown f(\bar{x})$,and $H(\bar{x})$.

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2e^{x_1^2+x_2^2}x_1 - 5 \\ \frac{\partial f}{\partial x_2} &= 2e^{x_1^2+x_2^2}x_2 + 10 \\ H_f &= \begin{pmatrix} 4e^{x_1^2+x_2^2}x_1^2 + 2e^{x_1^2+x_2^2} & 4e^{x_1^2+x_2^2}x_1x_2 \\ 4e^{x_1^2+x_2^2}x_1x_2 & 4e^{x_1^2+x_2^2}x_2^2 + 2e^{x_1^2+x_2^2} \end{pmatrix} \end{aligned}$$

So the quadratic approximation of $f(0, 1)$ is:

$$\begin{aligned} A &= (e + 10) + (-5, 2e + 10) \begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix} + \frac{1}{2}(x_1, x_2 - 1) \begin{pmatrix} 2e & 0 \\ 0 & 6e \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix} \\ &= e(x_1^2 + 3x_2^2 - 4x_2 + 2) + 10x_2 - 5x_1 \end{aligned}$$

In order to determine the approximation of f, we computing the Hessian matrix of equation A,

$$H_A = \begin{pmatrix} 2e & 0 \\ 0 & 6e \end{pmatrix}$$

So it's convex function considering the $D_1 > 0, D_2 > 0$.

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Problem 5

Consider the function $f : E_3 \to E_1$ given by $f(x) = x^T Ax$, where **(8 pts)**

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 2 \\ 1 & 1 & \theta \end{pmatrix}$$

a. Is A symmetric?

No, A is not symmetric.

b. What is the Hessian of f?*(Hint: change A into a symmetric matrix)*

$$H_f = (A + A')/2 = \begin{pmatrix} 2 & 1.5 & 2 \\ 1.5 & 3 & 1.5 \\ 2 & 1.5 & \theta \end{pmatrix}$$

c. For what value of θ is f strictly convex?

The condition for strictly convex, is $D_1 = 2 > 0, D_2 = 3.75 > 0$,
$D_3 = 2(3\theta - 1.5^2) - 1.5(1.5\theta - 3) + 2(1.5^2 - 6) > 0$, we get that $\theta > 2$.

Problem 6

Determine a direction of decrease of the function $f(x_1, x_2) = (x_1 - x_2^2)^2 + (1 - x_1)^2$ at the point $\bar{x} = (0, 1)$. Show that the direction you find is a decreasing direction. **(8 pts)**

First, computing the gradient of function $f(x_1, x_2) = x_2^4 + 2x_1^2 - 2x_1x_2^2 - x_1 + 1$.

$$\triangledown f(x)^T = (4x_1 - 2x_2^2 - 1, 4x_2^3 - 4x_1x_2)$$

Giving $d = (3, -4)$, the $\triangledown f(0, 1)^T \cdot d < 0$. So direction $(3, -4)$ is a decreasing direction.

Problem 7

Name an important property of convex functions. What is the significance of convexity in minimization? **(8 pts)**

If function f is convex function, then we can use H_f to determine the decreasing direction of function to get it's minimal point; It can reducing the complicated computing in the process of minimization.

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Problem 8

A function $f(x)$ of one variable is known to be twice differentiable. The following information is given: **(8 pts)**

$$f'(1) = 2, f''(1) = -3$$

Using Newtons Method, what is the next estimate of the location of the unconstrained minimizer?
For Newton's Method $x^{k+1} = x^k - \frac{f(x^k)'}{f(x^k)''}$ So the next estimate of the location of the unconstrained minimizer is $\frac{5}{3}$.

Problem 9

Suppose we are minimizing a function $f(x)$ over an interval $[a, b]$ and we have a point satisfying the sufficiency conditions for a local minimum. Suggest at least two ways of verifying that it is a global minimum. **(8 pts)**

a. To verify whether function f is convex function, if it's then local minimum is a global minimum.

b. Compare the function value of this point with $f(a), f(b)$, if it's the smallest,then this point is also a global minimum.

Problem 10

Answer True (T) or False (F). No explanation is necessary. **(8 pts)**

(a) The logarithmic function $\log(x)$ is convex?

F

(b) Product of two convex functions is also a convex function?

F

(c) A stationary point $\bar{x}$ of a single variable function $f(x)$ is a local maximum if $f''(\bar{x}) \leq 0$?

F

(d) If $f(a)f(b) < 0$ then $f(x)$ (a single variable function) has a zero point $\bar{x}$ between a and b ($f(\bar{x}) = 0$)?

F